

University of Groningen

## The Schur algorithm for generalized Schur functions III

Alpay, Daniel; Azizov, Tomas; Dijksma, Aad; Langer, Heinz

*Published in:*  
Linear Algebra and Its Applications

*DOI:*  
[10.1016/S0024-3795\(02\)00734-6](https://doi.org/10.1016/S0024-3795(02)00734-6)

**IMPORTANT NOTE:** You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2003

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Alpay, D., Azizov, T., Dijksma, A., & Langer, H. (2003). The Schur algorithm for generalized Schur functions III: J-unitary matrix polynomials on the circle. *Linear Algebra and Its Applications*, 369, 113-144. [https://doi.org/10.1016/S0024-3795\(02\)00734-6](https://doi.org/10.1016/S0024-3795(02)00734-6)

**Copyright**

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

**Take-down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

LINEAR ALGEBRA  
AND ITS  
APPLICATIONS

Linear Algebra and its Applications 369 (2003) 113–144

[www.elsevier.com/locate/laa](http://www.elsevier.com/locate/laa)

# The Schur algorithm for generalized Schur functions III: $J$ -unitary matrix polynomials on the circle

Daniel Alpay <sup>a</sup>, Tomas Azizov <sup>b</sup>, Aad Dijksma <sup>c,\*</sup>,  
Heinz Langer <sup>d</sup>

<sup>a</sup>*Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, 84105 Beer Sheva, Israel*

<sup>b</sup>*Department of Mathematics, Voronezh State University, 394693 Voronezh, Russia*

<sup>c</sup>*Department of Mathematics, University of Groningen, P.O. Box 800, Groningen 9700 AV, The Netherlands*

<sup>d</sup>*Institute of Analysis and Technical Mathematics, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria*

Received 10 June 2002; accepted 1 December 2002

Submitted by L. Rodman

## Abstract

The main result is that for

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

every  $J$ -unitary  $2 \times 2$ -matrix polynomial on the unit circle is an essentially unique product of elementary  $J$ -unitary  $2 \times 2$ -matrix polynomials which are either of degree 1 or  $2k$ . This is shown by means of the generalized Schur transformation introduced in [Ann. Inst. Fourier 8 (1958) 211; Ann. Acad. Sci. Fenn. Ser. A I 250 (9) (1958) 1–7] and studied in [Pisot and Salem Numbers, Birkhäuser Verlag, Basel, 1992; Philips J. Res. 41 (1) (1986) 1–54], and also in the first two parts [Operator Theory: Adv. Appl. 129, Birkhäuser Verlag, Basel, 2000, p. 1; Monatshefte für Mathematik, in press] of this series. The essential tool in this paper are the reproducing kernel Pontryagin spaces associated with generalized Schur functions.  
© 2003 Elsevier Science Inc. All rights reserved.

\* Corresponding author.

*E-mail addresses:* [dijksma@math.rug.nl](mailto:dijksma@math.rug.nl) (A. Dijksma); [dany@math.bgu.ac.il](mailto:dany@math.bgu.ac.il) (D. Alpay); [azizov@tom.vsu.ru](mailto:azizov@tom.vsu.ru) (T. Azizov); [hlanger@mail.zserv.tuwien.ac.at](mailto:hlanger@mail.zserv.tuwien.ac.at) (H. Langer).

AMS classification: 47A57; 46C20; 46E22; 47B50

Keywords: Generalized Schur algorithm; Generalized Schur functions; Kernels with negative squares; Reproducing kernel Pontryagin spaces; Elementary  $J$ -unitary matrix polynomials; Minimal factorizations

## 1. Introduction

A *generalized Schur function*  $s$  is a scalar-valued meromorphic function  $s$  on the open unit disk  $\mathbb{D}$  for which the kernel

$$S_s(z, w) := \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \mathcal{D}(s), \quad (1.1)$$

has a finite number of negative squares; here  $\mathcal{D}(s)$  denotes the domain of holomorphy of  $s$  in  $\mathbb{D}$ . If this number is  $\kappa$ ,  $s$  is said to be in the *generalized Schur class*  $\mathbf{S}_\kappa$ .

Recall the definition: Let  $\Omega$  be some set and  $m$  an integer  $\geq 1$ . A kernel  $K$  on  $\Omega$ , that is, a function  $K: \Omega \times \Omega \rightarrow \mathbb{C}^{m \times m}$ , is said to have  $\kappa$  *negative squares* on  $\Omega$  if it is *hermitian*:  $K(z, w)^* = K(w, z)$ ,  $z, w \in \Omega$ , and if for every integer  $r \geq 1$ , vectors  $\mathbf{c}_1, \dots, \mathbf{c}_r \in \mathbb{C}^m$ , and points  $w_1, \dots, w_r \in \Omega$ , the hermitian  $r \times r$  matrix with  $\ell, j$ -entry equal to  $\mathbf{c}_j^* K(w_j, w_\ell) \mathbf{c}_\ell$  has at most  $\kappa$  negative eigenvalues and exactly  $\kappa$  negative eigenvalues (counted with multiplicities) for some choice of  $r, \mathbf{c}_1, \dots, \mathbf{c}_r, w_1, \dots, w_r$ . If  $\kappa = 0$  the kernel  $K$  is said to be *positive* on  $\Omega$ . In the sequel, only the cases  $m = 1$  (as for the kernel in (1.1)) and  $m = 2$  are of interest. Kernels with negative squares were introduced by Krein in [18].

The class  $\mathbf{S}_0$  consists of all functions which are analytic and of modulus  $\leq 1$  in  $\mathbb{D}$ , that is, the class of all Schur functions. We denote by  $\mathbf{S}$  the union of all classes  $\mathbf{S}_\kappa$  for  $\kappa \geq 0$ , and by  $\mathbf{S}^0$  the subclass of those functions of  $\mathbf{S}$  which are analytic at  $z = 0$ . The classical Schur transformation, which associates with a function  $s \in \mathbf{S}_0$  a new function  $s_1$  by the formula

$$s_1(z) = \frac{1}{z} \frac{s(z) - s(0)}{1 - s(z)s(0)^*}, \quad (1.2)$$

was generalized in [13,14] (see also [11,15]) to functions of the class  $\mathbf{S}^0$ . This transformation is more complicated than (1.2) and takes different forms depending on the value of  $|s(0)|$  (see [2,11,13], and also Section 2 below). In the first two parts of this series this generalized Schur transformation for functions of the class  $\mathbf{S}^0$  was studied within the framework of realizations of analytic functions in Pontryagin spaces. In fact, with the function  $s \in \mathbf{S}^0$  a coisometric colligation in some Pontryagin space was associated, and the connections between the colligations corresponding to  $s$  and its Schur transformation  $s_1$  were studied. For the definition of these realizations via linear operators the analyticity of the function  $s$  at  $z = 0$  is necessary.

In the present paper we introduce the generalized Schur transformation  $s_1$  for all functions  $s$  of the class  $\mathbf{S}$  which are not constant of modulus one, including also those which have a pole at  $z = 0$  (see the formulas in Section 2), and we study the

corresponding reproducing kernel Pontryagin spaces which are associated with a function  $s$  and its generalized Schur transformation  $s_1$ . It turns out that these generalized Schur transformations can be described as fractional linear transformations whose four coefficients make up special polynomial  $2 \times 2$ -matrix functions which are  $J$ -unitary on the unit circle  $\mathbb{T}$  for the matrix

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.3)$$

We denote the class of all polynomial  $2 \times 2$ -matrix functions, which are  $J$ -unitary on  $\mathbb{T}$ , by  $\mathcal{U}_J$ . As one of the main results of this paper it is shown that the matrix functions which describe the generalized Schur transformation are the elementary factors of the class  $\mathcal{U}_J$  and that each matrix function of the class  $\mathcal{U}_J$  can be written in an essentially unique way as a product of such elementary factors. If the matrix function  $\Theta \in \mathcal{U}_J$  is, in particular,  $J$ -inner such a representation of  $\Theta$  as a product of elementary factors follows from results of Potapov [19].

In the sequel, the elements of the class  $\mathbf{S}$  are often represented in a projective or homogeneous way. Instead of the function  $s \in \mathbf{S}$  we consider pairs  $(u, v)$  of functions  $u, v$  which are analytic in  $\mathbb{D}$ , do not have common zeros, in particular,

$$|u(0)| + |v(0)| \neq 0,$$

and are such that  $s(z) = v(z)/u(z)$ . By  $\mathbf{S}^{\text{hom}}$  we denote the class of all such pairs  $(u, v)$  for which the Schur kernel

$$S_{u,v}(z, w) := \frac{u(z)u(w)^* - v(z)v(w)^*}{1 - zw^*}, \quad z, w \in \mathbb{D}, \quad (1.4)$$

has a finite number of negative squares; the set of such pairs for which this number of negative squares is  $\kappa$  is denoted by  $\mathbf{S}_\kappa^{\text{hom}}$ .

The outline of the paper is as follows. In Section 2 we define the generalized Schur transformation for functions of the class  $\mathbf{S}$  which are not constants of modulus one. The difference with the corresponding definitions in the papers [2,3] (taken from [11]) lies in the fact that here we define the transformation for all functions of  $\mathbf{S}$ , and it becomes a little simpler since poles at zero are allowed. In Section 3 we recall the definition of reproducing kernel Pontryagin spaces and some related facts needed in the sequel. In Section 4 the class  $\mathcal{U}_J$  of all polynomial  $2 \times 2$ -matrix functions  $\Theta$  which are  $J$ -unitary on the unit circle is studied, the corresponding finite-dimensional reproducing kernel Pontryagin spaces  $\mathcal{P}(\Theta)$  are considered and connections between factorizations of  $\Theta$  in  $\mathcal{U}_J$  and non-trivial invariant subspaces of the backward shift operator  $R_0$  in  $\mathcal{P}(\Theta)$  are established.

Sections 5 and 6 contain the main results of this paper. It is shown in Theorem 5.3 that there are two kinds of elementary factors in  $\mathcal{U}_J$ , and in Theorem 5.4 it is proved that each function in  $\mathcal{U}_J$  is an essentially unique product of these elementary factors. In Section 6 the corresponding reproducing kernel Pontryagin spaces are used in order to solve the following inverse problem (see Theorem 6.5): Given a pair  $(u, v) \in \mathbf{S}^{\text{hom}}$ , find all  $\Theta \in \mathcal{U}_J$  such that

$$(u(z) - v(z))\Theta(z) = z^{\deg \Theta}(\widehat{u}(z) - \widehat{v}(z)),$$

where  $(\widehat{u}, \widehat{v}) \in \mathbf{S}^{\text{hom}}$  and  $\deg \Theta$  stands for the McMillan degree of  $\Theta$  (the definition is recalled in Section 4). At the end of Section 6 it is shown how for a given function  $\Theta \in \mathcal{U}_J$  its representation as a product of elementary factors can be obtained by means of the generalized Schur algorithm. Finally, Section 7 contains some examples.

In a fourth part of this series analogous results for the case of the real line will be considered. Also the matrix version of the generalized Schur algorithm, partly treated in [1] and [6], will be considered elsewhere. In [4] we relate the generalized Schur transform defined in Section 2 to a basic interpolation problem in the class  $\mathbf{S}^0$ .

The authors thank Gerald Wanjala for careful reading of the manuscript and checking the calculations.

## 2. The generalized Schur algorithm

If  $s \in \mathbf{S}_\kappa$  is not a constant function of modulus one we define the *generalized Schur transformation* as follows:

(a<sub>+</sub>) If  $|s(0)| < 1$  then

$$s_1(z) = \frac{1}{z} \frac{s(z) - s(0)}{1 - s(z)s(0)^*}. \quad (2.1)$$

(a<sub>-</sub>) If  $|s(0)| > 1$  and  $s$  is analytic at zero then

$$s_1(z) = z \frac{1 - s(z)s(0)^*}{s(z) - s(0)}, \quad (2.2)$$

if  $z = 0$  is a pole of  $s$ , then

$$s_1(z) = z s(z). \quad (2.3)$$

(b) If  $|s(0)| = 1$  then define  $k$  by the Taylor expansion of  $s$  at 0:

$$s(z) = \sigma_0 + \sigma_k z^k + \sigma_{k+1} z^{k+1} + \dots, \quad \sigma_k \neq 0$$

and introduce numbers  $c_j$ ,  $j = 0, 1, \dots$ , by the relation

$$(s(z) - \sigma_0)(c_0 + c_1 z + \dots + c_n z^n + \dots) \equiv \sigma_0 z^k.$$

With the polynomial

$$Q(z) = c_0 + \dots + c_{k-1} z^{k-1} - (c_{k-1}^* z^{k+1} + \dots + c_0^* z^{2k})$$

the transformed function  $s_1(z)$  is

$$s_1(z) = \frac{(Q(z) - z^k)s(z) - \sigma_0 Q(z)}{\sigma_0^* Q(z)s(z) - (Q(z) + z^k)}. \quad (2.4)$$

The formula (2.1) is the classical Schur transformation. The cases  $(a_\pm)$  have been extended to matrix-valued generalized Schur functions in [1] and [6]. The formulas

in cases (a<sub>-</sub>) and (b) allow that the transformed function  $s_1$  has a pole in  $z = 0$ . This was excluded in [2,11,13–15]. Note that in case (b)

$$z^{2k} Q(1/z^*)^* = -Q(z).$$

**Lemma 2.1.** *The pairs  $(u, v)$ ,  $(\widehat{u}, \widehat{v}) \in \mathbf{S}_k^{\text{hom}}$  generate the same Schur kernel (1.4):*

$$S_{(u,v)}(w, z) = S_{(\widehat{u}, \widehat{v})}(w, z), \quad z, w \in \mathbb{D},$$

*if and only if  $(u - v)\Theta = (\widehat{u} - \widehat{v})$  with some constant  $J$ -unitary matrix  $\Theta$ .*

Recall that  $\Theta$  is a constant  $J$ -unitary  $2 \times 2$  matrix if and only if it is of the form

$$\Theta = \frac{1}{\sqrt{1 - |\rho|^2}} \begin{pmatrix} 1 & \rho \\ \rho^* & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

for complex numbers  $\rho \in \mathbb{D}$  and  $c_1, c_2$  with  $|c_1| = |c_2| = 1$  (see, for example, [16, Theorem 1.2]). A proof of Lemma 2.1 can be found in [7, Lemma 5.1].

The formulas (2.1)–(2.4) can be written in the homogeneous representation of the functions  $s$  and  $s_1$  as

$$z^\alpha (u_1(z) - v_1(z)) = (u(z) - v(z))\Theta(z), \quad (2.5)$$

where  $\alpha = 1$  in cases (a) and  $\alpha = 2k$  in case (b), and  $\Theta(z)$  is the matrix function

$$\Theta(z) = \begin{cases} \widehat{\Theta}_{a+}(z) := \begin{pmatrix} z|u(0)|^2 & v(0)u(0)^* \\ zu(0)v(0)^* & |u(0)|^2 \end{pmatrix} & \text{if } |v(0)| < |u(0)|, \\ \widehat{\Theta}_{a-}(z) := \begin{pmatrix} v(0)u(0)^* & z|u(0)|^2 \\ |u(0)|^2 & zu(0)v(0)^* \end{pmatrix} & \text{if } |v(0)| > |u(0)| > 0, \\ \widehat{\Theta}_{a-}(z) := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} & \text{if } u(0) = 0, \\ \widehat{\Theta}_b(z) := \begin{pmatrix} -(Q(z) + z^k)|u(0)|^2 & Q(z)v(0)u(0)^* \\ -Q(z)v(0)^*u(0) & (Q(z) - z^k)|u(0)|^2 \end{pmatrix} & \text{if } |v(0)| = |u(0)| > 0. \end{cases}$$

If we multiply these matrix functions by suitable numbers they become  $J$ -unitary on  $\mathbb{T}$ , and if we normalize them afterwards by the condition that  $\Theta(1) = I_2$  we obtain for example in the first case.

$$\begin{aligned} \Theta_{a+}(z) &:= \widehat{\Theta}_{a+}(z)\widehat{\Theta}_{a+}(1)^{-1} \\ &= I_2 - \frac{(1-z)}{|u(0)|^2 - |v(0)|^2} \begin{pmatrix} |u(0)|^2 & -u(0)^*v(0) \\ u(0)v(0)^* & -|v(0)|^2 \end{pmatrix}, \end{aligned}$$

and the same expression for  $\Theta_{a-}(z)$ . With the vector

$$\mathbf{u}_0 := \begin{pmatrix} u(0) \\ v(0) \end{pmatrix},$$

this formula becomes

$$\Theta_{a_{\pm}}(z) = \Theta_a(z) := I_2 - \frac{1-z}{\mathbf{u}_0^* J \mathbf{u}_0} \mathbf{u}_0 \mathbf{u}_0^* J.$$

For  $\widehat{\Theta}_b(z)$  we obtain by an analogous calculation

$$\begin{aligned} \Theta_b(z) &= z^k I_2 + \frac{Q(z) - z^k Q(1)}{|u(0)|^2} \begin{pmatrix} |u(0)|^2 & -u(0)^* v(0) \\ u(0) v(0)^* & -|u(0)|^2 \end{pmatrix} \\ &= z^k I_2 + \mathbf{u}_0 \mathbf{u}_0^* J Q_1(z), \end{aligned}$$

where

$$\mathbf{u}_0 := \begin{pmatrix} 1 \\ v(0)^*/u(0)^* \end{pmatrix}, \quad Q_1(z) := Q(z) - z^k Q(1). \quad (2.6)$$

We mention that if  $z^{2k} Q(1/z^*)^* = -Q(z)$  then also the function  $Q_1(z)$  in (2.6) has this property. Finally we note the formulas

$$\det \Theta_{a_{\pm}}(z) = z, \quad \det \Theta_b(z) = z^{2k}, \quad (2.7)$$

which follow from the above expressions for  $\Theta_a$  and  $\Theta_b$  and from the formula  $\det(I - AB) = \det(I - BA)$ , which holds for any pair of matrices  $A, B$  of appropriate dimensions.

### 3. Reproducing kernel Pontryagin spaces

For the general theory of reproducing kernel Pontryagin spaces we refer to [5] and [21]. We denote the negative index of a Pontryagin space  $\mathcal{P}$  by  $\text{ind}_- \mathcal{P}$ ; by definition this number is finite.

A kernel  $K: \Omega \times \Omega \rightarrow \mathbb{C}^{m \times m}$ , which is positive on a set  $\Omega$ , is of finite rank  $r$  if it can be written as  $K(z, w) = \sum_{j=1}^r \mathbf{a}_j(z) \mathbf{a}_j(w)^*$  with linearly independent  $m$ -vector-valued functions  $\mathbf{a}_j$  in  $\Omega$ ,  $j = 1, 2, \dots, r$ . For the following result we refer to [20,21].

**Proposition 3.1.** *A kernel  $K$  on a set  $\Omega$  has a finite number  $\kappa$  of negative squares if and only if it can be written as a difference  $K(z, w) = K_+(z, w) - K_-(z, w)$ ,  $z, w \in \Omega$ , where  $K_+$  and  $K_-$  are positive kernels on  $\Omega$  and  $K_-$  has finite rank. In this case,  $K_-$  can be chosen to be of rank  $\kappa$  and such that the intersection of the reproducing kernel Hilbert spaces with reproducing kernels  $K_+$  and  $K_-$  is only the zero function. Moreover, there exists a uniquely defined reproducing kernel Pontryagin space  $\mathcal{P}(K)$  of  $\mathbb{C}^m$ -valued functions defined on  $\Omega$  with reproducing kernel  $K$ , and  $\text{ind}_- \mathcal{P}(K) = \kappa$ .*

Sometimes we use different notations such as  $\mathcal{P}(s)$  instead of  $\mathcal{P}(S_s)$  for the Schur kernel  $S_s$ . These will be understood from the context.

Let us recall that the Pontryagin space  $\mathcal{P}(K)$  is characterized by the following two properties:

- (1) For every  $w \in \Omega$  and  $\mathbf{c} \in \mathbb{C}^m$ , the function  $z \mapsto K(z, w)\mathbf{c}$  belongs to  $\mathcal{P}(K)$ .
- (2) For every  $\mathbf{f} \in \mathcal{P}(K)$ ,  $w \in \Omega$ ,  $\mathbf{c} \in \mathbb{C}^m$  it holds

$$\langle \mathbf{f}, K(\cdot, w)\mathbf{c} \rangle_{\mathcal{P}(K)} = \mathbf{c}^* \mathbf{f}(w).$$

We consider some examples of reproducing kernel Pontryagin spaces.

**Example 3.2.** With a function  $s \in \mathbf{S}_\kappa$  we associate the Schur kernel (1.1)

$$K(z, w) = S_s(z, w) := \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \mathcal{D}(s).$$

It generates a reproducing kernel Pontryagin space  $\mathcal{P}(s)$  with  $\text{ind}_- \mathcal{P}(s) = \kappa$ , which is the closed linear span of the functions  $S_s(\cdot, w)$ ,  $w \in \mathcal{D}(s)$ , equipped with the inner product generated by

$$\langle S_s(\cdot, w), S_s(\cdot, z) \rangle_{\mathcal{P}(s)} = S_s(z, w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \mathcal{D}(s).$$

For  $\kappa = 0$  this space  $\mathcal{P}(s)$  is the de Branges–Rovnyak space of the Schur function  $s$ , which is a reproducing kernel Hilbert space. If, in particular,  $s \in \mathbf{S}_0$  is inner (that is,  $|s(z)| = 1$  for almost all  $z \in \mathbb{T}$ ) then we have

$$\mathcal{P}(s) = \mathbf{H}_2 \ominus s\mathbf{H}_2$$

(see [16]). For general  $s \in \mathbf{S}_0$  the space  $\mathcal{P}(s)$  is contractively included in the Hardy space  $\mathbf{H}_2$  of the unit disk. We recall that  $\mathbf{H}_2$  is the reproducing kernel Hilbert space with reproducing kernel  $1/(1 - zw^*)$ .

If  $\kappa > 0$ , according to [17], the function  $s$  can be written as

$$s(z) = \frac{s_0(z)}{b(z)}, \quad (3.1)$$

with  $s_0 \in \mathbf{S}_0$  and a Blaschke product  $b$  of degree  $\kappa$ , such that  $s_0$  and  $b$  do not have common zeros. Recall that a Blaschke product  $b$  of degree  $\kappa$  is a function of the form

$$b(z) = \prod_{j=1}^{\kappa} \frac{z - \alpha_j}{1 - z\alpha_j^*},$$

with not necessarily distinct numbers  $\alpha_1, \alpha_2, \dots, \alpha_\kappa \in \mathbb{D}$ . It follows that

$$\frac{1 - s(z)s(w)^*}{1 - zw^*} = \frac{1}{b(z)b(w)^*} \left\{ \frac{1 - s_0(z)s_0(w)^*}{1 - zw^*} - \frac{1 - b(z)b(w)^*}{1 - zw^*} \right\},$$

and the associated reproducing kernel Pontryagin space  $\mathcal{P}(s)$  can be characterized as

$$\mathcal{P}(s) = \left\{ F(z) = \frac{f_0(z) + f(z)}{b(z)} \mid f_0 \in \mathcal{P}(s_0), f \in \mathcal{P}(b) \right\},$$



with the inner product given by

$$\langle F, F \rangle_{\mathcal{P}(s)} = \langle f_0, f_0 \rangle_{\mathcal{P}(s_0)} - \langle f, f \rangle_{\mathcal{P}(b)}.$$

This corresponds to the decomposition of Proposition 3.1:

$$\frac{1 - s(z)s(w)^*}{1 - zw^*} = K_+(z, w) - K_-(z, w),$$

where

$$K_+(z, w) = \frac{1}{b(z)b(w)^*} \frac{1 - s_0(z)s_0(w)^*}{1 - zw^*},$$

$$K_-(z, w) = \frac{1}{b(z)b(w)^*} \frac{1 - b(z)b(w)^*}{1 - zw^*}.$$

We will also use a homogeneous representation of the space  $\mathcal{P}(s)$ . To this end we represent the function  $s$  as a quotient of two functions  $u, v \in \mathbf{H}_2$ , such that  $|u(0)| + |v(0)| \neq 0$ :  $s(z) = v(z)/u(z)$ . For example, we can choose  $u(z) = b(z)$ ,  $v(z) = s_0(z)$  from (3.1). Recall from Section 1 (see (1.4)) that the class  $\mathbf{S}_\kappa^{\text{hom}}$  consists of all such pairs  $(u, v)$  of functions on  $\mathbb{D}$  for which the kernel

$$S_{(u,v)}(z, w) = \frac{u(z)u(w)^* - v(z)v(w)^*}{1 - zw^*}, \quad z, w \in \mathbb{D}, \quad (3.2)$$

has  $\kappa$  negative squares. The corresponding reproducing kernel space  $\mathcal{P}(u, v)$ , which can be considered to be the closed linear span of the functions  $S_{(u,v)}(\cdot, w)$ ,  $w \in \mathbb{D}$ , equipped with the inner product

$$\langle S_{(u,v)}(\cdot, w), S_{(u,v)}(\cdot, z) \rangle_{\mathcal{P}(u,v)} := S_{(u,v)}(z, w),$$

is a Pontryagin space with negative index  $\kappa$ . If for a given function  $s \in \mathbf{S}_\kappa$  and a pair  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$  we have  $s(z) = v(z)/u(z)$ ,  $z \in \mathcal{D}(s)$ , then the mapping  $T$  defined by

$$(Th)(z) = u(z)h(z), \quad z \in \mathcal{D}(s), \quad h \in \mathcal{P}(s),$$

establishes an isomorphism between the Pontryagin spaces  $\mathcal{P}(s)$  and  $\mathcal{P}(u, v)$  (see [5, Theorem 1.5.7]).

**Example 3.3.** Kernels of the form (1.1) can have a finite number of negative squares for functions  $s$  which are not holomorphic at  $z = 0$ . We recall the example from [5, p. 82]: Consider the function  $s(z) = \delta_0(z)$ , where  $\delta_\alpha(z) = 1$  if  $z = \alpha$  and  $z = 0$  if  $z \neq \alpha$ . The associated kernel (1.1) can be written as

$$\begin{aligned} \frac{1 - s(z)s(w)^*}{1 - zw^*} &= \frac{1 - \delta_0(z)\delta_0(w)^*}{1 - zw^*} \\ &= \frac{1}{1 - zw^*} - \frac{\delta_0(z)\delta_0(w)^*}{1 - zw^*} \\ &= \frac{1}{1 - zw^*} - \delta_0(z)\delta_0(w), \end{aligned}$$

It is therefore the difference of a positive kernel and a positive kernel of rank 1. The associated reproducing kernel Pontryagin space has negative index 1 and consists of all functions of the form  $h(z) = f(z) + c\delta_0(z)$  where  $f \in \mathbf{H}_2$  and  $c \in \mathbb{C}$  with the inner product given by

$$\langle h, h \rangle = \|f\|_{\mathbf{H}_2}^2 - |c|^2.$$

More generally, the kernel

$$K(z, w) := \frac{1}{1 - zw^*} - \sum_1^\kappa \delta_{w_j}(z)\delta_{w_j}(w), \quad z, w \in \mathbb{D},$$

with mutually different points  $w_j \in \mathbb{D}$ ,  $j = 1, 2, \dots, r$ , has  $\kappa$  negative squares.

#### 4. $J$ -unitary polynomials

1. Recall that

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A  $J$ -unitary polynomial  $\Theta$  is by definition a  $\mathbb{C}^{2 \times 2}$ -valued polynomial which is  $J$ -unitary on the unit circle, that is,  $\Theta(z)^* J \Theta(z) = J$ ,  $z \in \mathbb{T}$ , but we shall mostly use the equivalent form

$$\Theta(z) J \Theta(z)^* = J, \quad z \in \mathbb{T}; \quad (4.1)$$

the set of all  $J$ -unitary polynomials is denoted by  $\mathcal{U}_J$ . If  $\Theta \in \mathcal{U}_J$ , by analytic continuation, from (4.1) we obtain

$$\Theta(z) J \Theta(1/\bar{z})^* = J, \quad z \in \mathbb{C}, \quad z \neq 0. \quad (4.2)$$

In particular,  $\det \Theta(z) \neq 0$  if  $z \neq 0$ . That  $\det \Theta(0)$  may be zero for  $\Theta \in \mathcal{U}_J$  is illustrated by the example

$$\Theta(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

The function  $\Theta \in \mathcal{U}_J$  is said to be  $J$ -inner if it satisfies

$$\Theta(z) J \Theta(z)^* \leq J, \quad z \in \mathbb{D}.$$

A  $J$ -unitary polynomial  $\Theta$  is in particular a  $\mathbb{C}^{2 \times 2}$ -valued rational function which is analytic at the origin, and as such it admits a *realization*, that is, a representation of the form

$$\Theta(z) = D + zC(I_N - zA)^{-1}B, \quad (4.3)$$

where  $(A, B, C, D) \in \mathbb{C}^{N \times N} \times \mathbb{C}^{N \times 2} \times \mathbb{C}^{2 \times N} \times \mathbb{C}^{2 \times 2}$  (see [10]). Such a realization and, in particular, the dimension  $N$  are not unique. If  $N$  is chosen minimal then

the realization is unique up to similarity and  $N$  is then called the *McMillan degree* of  $\Theta$  (see [10]), and here denoted by  $\deg \Theta$ . The McMillan degree has an equivalent (and more direct, albeit more complicated) definition in terms of zeros and poles (see [10, Section 4.2]). In case  $\Theta \in \mathcal{U}_J$  and  $\Theta(z) = \sum_{j=0}^n \Theta_j z^j$  this definition reads:

$$\deg \Theta = \text{rank} \begin{pmatrix} \Theta_n & \Theta_{n-1} & \cdots & \Theta_1 \\ 0 & \Theta_n & \cdots & \Theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Theta_n \end{pmatrix}.$$

A form of the realization of elements of  $\mathcal{U}_J$  is given in the next proposition (for a proof see [6] or, in a more general setting, [7, Theorem 4.1]).

**Proposition 4.1.** *Let  $(C, A) \in \mathbb{C}^{2 \times N} \times \mathbb{C}^{N \times N}$  be such that*

- (1)  *$A$  is nilpotent,*
- (2)  *$\bigcap_{\ell=0}^{\infty} \ker C A^\ell = \{0\}$ , and*
- (3) *the following matrix  $\mathbb{Q}$  is invertible:*

$$\mathbb{Q} = \sum_{\ell=0}^{\infty} (A^\ell)^* C^* J C A^\ell. \quad (4.4)$$

*Then for arbitrary  $w \in \mathbb{T}$  the function*

$$\Theta_w(z) = I - (1 - zw^*)C(I - zA)^{-1}\mathbb{Q}^{-1}(I - w^*A^*)^{-1}C^*J \quad (4.5)$$

*belongs to  $\mathcal{U}_J$ . Conversely, every function  $\Theta \in \mathcal{U}_J$  is of this form. For a given pair  $(C, A)$  with the above properties and for any two points  $v, w \in \mathbb{T}$  the functions  $\Theta_v(z)$  and  $\Theta_w(z)$  differ by a constant  $J$ -unitary right factor.*

We note that both the intersection  $\bigcap_{\ell=0}^{\infty} \ker C A^\ell$  and the sum (4.4) are finite since  $A$  is nilpotent. The realization (4.5) is not of the form (4.3) but has the property  $\Theta_w(w) = I$ . To reduce it to the form (4.3) it suffices to observe that

$$(1 - zw^*)(I_N - zA)^{-1} = I_N - z(w^*I_N - A)(I_N - zA)^{-1}.$$

2. By  $\mathbf{H}_{2,J}$  we denote the space  $\mathbf{H}_2^2 = \mathbf{H}_2 \oplus \mathbf{H}_2$  endowed with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{H}_{2,J}} := \langle \mathbf{f}, J\mathbf{g} \rangle_{\mathbf{H}_2^2}. \quad (4.6)$$

It is a Kreĭn space. For  $\Theta \in \mathcal{U}_J$  the operator of multiplication by  $\Theta$  from the left is a continuous isometry from  $\mathbf{H}_{2,J}$  into itself. We associate with  $\Theta \in \mathcal{U}_J$  the space

$$\mathcal{P}(\Theta) := \mathbf{H}_{2,J} \ominus \Theta \mathbf{H}_{2,J}. \quad (4.7)$$

The main properties of the space  $\mathcal{P}(\Theta)$ , which are relevant for the present paper, are collected in the following theorem. Item (1a) in the theorem shows that the notation  $\mathcal{P}(\Theta)$  is consistent with the notation introduced earlier.

**Theorem 4.2**

(1) For  $\Theta \in \mathcal{U}_J$  the following statements are true.

(1a)  $\mathcal{P}(\Theta)$  is the reproducing kernel Pontryagin space with reproducing kernel

$$K_\Theta(z, w) = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}, \quad z, w \in \mathbb{D}, \quad (4.8)$$

and hence the negative (resp. positive) index of  $\mathcal{P}(\Theta)$  equals the number of negative (resp. positive) squares of the kernel  $K_\Theta$  on  $\mathbb{D}$ .

(1b)  $\dim \mathcal{P}(\Theta)$  is finite and equal to the McMillan degree of  $\Theta$ .

(1c)  $\mathcal{P}(\Theta)$  is a Hilbert space if and only if the kernel  $K_\Theta$  is positive on  $\mathbb{D}$  or, equivalently, if and only if  $\Theta$  is  $J$ -inner.

(1d) The elements of  $\mathcal{P}(\Theta)$  are  $\mathbb{C}^2$ -valued polynomials.

(1e) The space  $\mathcal{P}(\Theta)$  is invariant under the backward-shift operators  $R_a$  defined by

$$R_a \mathbf{f}(z) = \frac{\mathbf{f}(z) - \mathbf{f}(a)}{z - a}, \quad a \in \mathbb{C}.$$

(1f) For all  $\mathbf{c} \in \mathbb{C}^2$ ,

$$(R_a \Theta)(z) \mathbf{c} = \frac{\Theta(z) - \Theta(a)}{z - a} \mathbf{c} \in \mathcal{P}(\Theta).$$

(2) Any finite dimensional non-trivial non-degenerate  $R_0$ -invariant subspace of  $\mathbf{H}_{2,J}$ , whose elements are polynomials, is of the form  $\mathcal{P}(\Theta)$  for some  $\Theta \in \mathcal{U}_J$ .

The first three statements remain true for any signature matrix  $J$  and any  $J$ -unitary rational function, (1e) and (1f) remain true for any  $J$ -unitary rational function if  $a$  is chosen to be a point of analyticity of  $\Theta$ ; compare [8]. Statement (2) of the theorem is the finite dimensional version of a result of Ball and Helton [9], de Branges [12], and Dym [16].

**Proof of (1a)–(1f).** First we note that by (4.2) the kernel in (4.1) can be written as

$$K_\Theta(z, w) = \begin{cases} J - \Theta(z)J\Theta(0)^*, & w = 0, \\ \frac{\Theta(\frac{1}{w^*}) - \Theta(z)}{\frac{1}{w^*} - z} \cdot \frac{J\Theta(w)^*}{w^*}, & w \neq 0, \end{cases}$$

from which it follows that  $K_\Theta(z, w)\mathbf{c}$  is a  $\mathbb{C}^2$ -valued polynomial in  $z$  for every  $w \in \mathbb{C}$  and  $\mathbf{c} \in \mathbb{C}^2$  and the degree of these polynomials is uniformly bounded by the degree of  $\Theta(z)$  as a polynomial in  $z$ . The first implies that  $K_\Theta(\cdot, w)\mathbf{c} \in \mathbf{H}_{2,J}$  and the second that the span of these polynomials is finite dimensional. Now we follow the proof of Theorem 2.8 of [16]: For  $\mathbf{c} \in \mathbb{C}^2$  and  $\mathbf{u} \in \mathbf{H}_{2,J}$ , we have

$$\begin{aligned}
\langle K_{\Theta}(z, w)\mathbf{c}, \Theta(z)\mathbf{u}(z) \rangle_{\mathbf{H}_{2,J}} &= \left\langle \frac{J\mathbf{c}}{1-zw^*}, \Theta(z)\mathbf{u}(z) \right\rangle_{\mathbf{H}_{2,J}} \\
&\quad - \left\langle \frac{\Theta(z)J\Theta(w)^*\mathbf{c}}{1-zw^*}, \Theta(z)\mathbf{u}(z) \right\rangle_{\mathbf{H}_{2,J}} \\
&= \left\langle \frac{J\mathbf{c}}{1-zw^*}, \Theta(z)\mathbf{u}(z) \right\rangle_{\mathbf{H}_{2,J}} \\
&\quad - \left\langle \frac{J\Theta(w)^*\mathbf{c}}{1-zw^*}, \mathbf{u}(z) \right\rangle_{\mathbf{H}_{2,J}} \\
&= \left\langle \frac{\mathbf{c}}{1-zw^*}, \Theta(z)\mathbf{u}(z) \right\rangle_{\mathbf{H}_2^2} - \left\langle \frac{\Theta(w)^*\mathbf{c}}{1-zw^*}, \mathbf{u}(z) \right\rangle_{\mathbf{H}_2^2} \\
&= \mathbf{u}(w)^*\Theta(w)^*\mathbf{c} - \mathbf{u}(w)^*\Theta(w)^*\mathbf{c} = 0.
\end{aligned}$$

The second equality follows from the fact that multiplication by  $\Theta$  is an isometry on  $\mathbf{H}_{2,J}$ . For the third equality we used the inner product (4.6) and for the fourth equality that  $(1/(1-zw^*))$  is the reproducing kernel of  $\mathbf{H}_2$ . The calculation shows that

$$K_{\Theta}(\cdot, w)\mathbf{c} \in \mathbf{H}_{2,J} \ominus \Theta\mathbf{H}_{2,J}.$$

For  $\mathbf{f} \in \mathbf{H}_{2,J} \ominus \Theta\mathbf{H}_{2,J}$  and  $\mathbf{c} \in \mathbb{C}^2$  we get, since

$$\begin{aligned}
\Theta(z)\frac{J\Theta(w)^*\mathbf{c}}{1-zw^*} &\in \Theta\mathbf{H}_{2,J}, \\
\langle \mathbf{f}, K_{\Theta}(\cdot, w)\mathbf{c} \rangle_{\mathbf{H}_{2,J}} &= \left\langle \mathbf{f}(z), \frac{J\mathbf{c}}{1-zw^*} \right\rangle_{\mathbf{H}_{2,J}} - \left\langle \mathbf{f}(z), \Theta(z)\frac{J\Theta(w)^*\mathbf{c}}{1-zw^*} \right\rangle_{\mathbf{H}_{2,J}} \\
&= \mathbf{c}^*\mathbf{f}(w).
\end{aligned}$$

Hence  $K_{\Theta}(z, w)$  is the reproducing kernel of  $\mathbf{H}_{2,J} \ominus \Theta\mathbf{H}_{2,J}$  making it a reproducing kernel Krein space. In particular the span of the functions  $K_{\Theta}(\cdot, w)\mathbf{c}$  with  $w \in \mathbb{C}$  and  $\mathbf{c} \in \mathbb{C}^2$  is dense in this space and, since this span is finite dimensional,  $\mathbf{H}_{2,J} \ominus \Theta\mathbf{H}_{2,J}$  is a finite dimensional Pontryagin space and its elements are  $\mathbb{C}^2$ -valued polynomials in  $z$ . This proves (1d), the first part of (1b) and the first statement in (1a). The statements concerning the indices of  $\mathcal{P}(\Theta)$  follow from, for example, [5, Theorem 1.1.2]. The McMillan degree of  $\Theta(z)$  is invariant under Möbius transformations (see [10, p. 83]). Under such a transformation  $\Theta(z)$  becomes a rational function, which is  $J$ -unitary on the unit circle and analytic and invertible at  $\infty$ , and then the equality  $\dim \mathcal{P}(\Theta) = \deg \Theta$  follows from [8, Theorem 6.1]. The first statement in (1c) follows from (1a); the if part of the second statement follows from [16, Theorem 2.6] and the only if part follows from  $K_{\Theta}(z, z) \geq 0$ . Finally, (1e) and (1f) follow from, for example, [5, Section 3.2] (see, in particular, formula (3.2.6)).  $\square$

**Remark 4.3.** The following observation will be used in the sequel. In the situation of Theorem 4.2 (2), the finite dimensionality and the  $R_0$ -invariance of this subspace imply that  $R_0$  has at least one eigenvalue, say  $\lambda_0$ , and a corresponding eigenvector, say  $\mathbf{f}_0$ . The equation  $R_0\mathbf{f}_0 = \lambda_0\mathbf{f}_0$  leads to  $\mathbf{f}_0(z) = \mathbf{f}_0(0)/(1 - \lambda_0 z)$ . Since the elements of  $\mathcal{P}(\Theta)$  are polynomials it follows that  $\lambda_0 = 0$  and therefore  $\mathbf{f}_0(z)$  is constant,  $\mathbf{f}_0(z) = \mathbf{u}$  with  $\mathbf{u} \in \mathbb{C}^2$ . In particular,  $\sigma(R_0) = \{0\}$ .

**Example 4.4.** The function  $\Theta(z) = zI_2$  belongs to  $\mathcal{U}_J$  and  $\mathcal{P}(\Theta)$  is the space  $\mathbb{C}^2$  equipped with the inner product  $\mathbf{c}^* J \mathbf{b}$ ,  $\mathbf{b}, \mathbf{c} \in \mathbb{C}^2$ . Indeed,

$$\mathcal{P}(\Theta) = \text{span}\{K_\Theta(\cdot, w)\mathbf{c} \mid \mathbf{c} \in \mathbb{C}^2\}$$

and for  $\mathbf{c} \in \mathbb{C}^2$ ,  $z, w \in \mathbb{D}$ , we have

$$K_\Theta(z, w)\mathbf{c} = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}\mathbf{c} = J\mathbf{c}.$$

Thus we may identify this function with the vector  $\mathbf{c}$ . The definition of the inner product in  $\mathcal{P}(\Theta)$  yields

$$\langle K_\Theta(\cdot, w)\mathbf{b}, K_\Theta(\cdot, z)\mathbf{c} \rangle_{\mathcal{P}(\Theta)} = \mathbf{c}^* K_\Theta(z, w)\mathbf{b} = \mathbf{c}^* J \mathbf{b}, \quad \mathbf{b}, \mathbf{c} \in \mathbb{C}^2,$$

which shows that the identification is an isomorphism. More generally, for  $n \in \mathbb{N}$  the function  $\Theta(z) = z^n I_2$  belongs to  $\mathcal{U}_J$  and, compare with (4.7),

$$\mathcal{P}(\Theta) = \left\{ \mathbf{f}(z) = \sum_{j=0}^{n-1} \mathbf{c}_j z^j : \mathbf{c}_j \in \mathbb{C}^2 \right\} = \mathbf{H}_{2,J} \ominus z^n \mathbf{H}_{2,J}.$$

3. In this subsection we prove some general results about factorizations of functions of the class  $\mathcal{U}_J$ .

**Proposition 4.5.** If  $\Theta_1, \Theta_2 \in \mathcal{U}_J$ , then  $\Theta = \Theta_1 \Theta_2 \in \mathcal{U}_J$  and

$$\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1) \oplus \Theta_1 \mathcal{P}(\Theta_2), \quad (4.9)$$

$$\deg \Theta = \deg \Theta_1 + \deg \Theta_2. \quad (4.10)$$

**Proof.** We have the decomposition

$$K_{\Theta_1 \Theta_2}(z, w) = K_{\Theta_1}(z, w) + \Theta_1(z) K_{\Theta_2}(z, w) \Theta_1(w)^*. \quad (4.11)$$

Since  $\mathcal{P}(\Theta_2) \subset \mathbf{H}_{2,J}$ , the reproducing kernel Pontryagin space with reproducing kernel  $\Theta_1(z) K_{\Theta_2}(z, w) \Theta_1(w)^*$  is included in  $\Theta_1 \mathbf{H}_{2,J}$ . In view of (4.7) its intersection with  $\mathcal{P}(\Theta_1)$  is the zero element. By [5, Theorem 1.5.5 (3) and Theorem 1.5.3 (4)], the decomposition (4.11) yields the orthogonal decomposition (4.9). The equality (4.10) follows from (4.9) and Theorem 4.2 (1b).  $\square$

Condition (4.10) means that the factorization  $\Theta = \Theta_1 \Theta_2$  is *minimal*. The factorization  $\Theta = \Theta_1 \Theta_2$  is called *trivial* if at least one of the factors is a  $J$ -unitary

constant, and *non-trivial* otherwise. A  $J$ -unitary polynomial  $\Theta \in \mathcal{U}_J$  is said to be *elementary* if there is no non-trivial factorization  $\Theta = \Theta_1 \Theta_2$  with  $\Theta_1, \Theta_2 \in \mathcal{U}_J$ .

**Theorem 4.6.** *Let  $\Theta \in \mathcal{U}_J$ . There is a one-to-one correspondence between non-trivial factorizations  $\Theta = \Theta_1 \Theta_2$  with  $\Theta_1, \Theta_2 \in \mathcal{U}_J$  and non-degenerate non-trivial  $R_0$ -invariant subspaces of  $\mathcal{P}(\Theta)$ . Here we do not distinguish between the factorizations  $\Theta = \Theta_1 \Theta_2$  and  $\Theta = \Theta'_1 \Theta'_2$  if  $\Theta'_1 = \Theta_1 U$  and  $\Theta'_2 = U^{-1} \Theta_2$  for a constant  $J$ -unitary matrix  $U$ .*

**Proof.** Let  $\mathcal{M}_1$  be an  $R_0$ -invariant subspace of  $\mathcal{P}(\Theta)$  which is non-degenerate. By Theorem 4.2 (2), we have  $\mathcal{M}_1 = \mathcal{P}(\Theta_1)$  for some  $\Theta_1 \in \mathcal{U}_J$ . Define a space  $\mathcal{M}_2$  via

$$\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1) \oplus \Theta_1 \mathcal{M}_2,$$

with the indefinite inner product given by

$$\langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{M}_2} = \langle \Theta_1 \mathbf{f}, \Theta_1 \mathbf{f} \rangle_{\mathcal{P}(\Theta)}.$$

The space  $\mathcal{M}_2$  provided with this inner product is non-degenerate. We apply Theorem 4.2 (2) to prove that  $\mathcal{M}_2 = \mathcal{P}(\Theta_2)$  for some  $\Theta_2 \in \mathcal{U}_J$ :

Since  $\Theta_1 \mathcal{M}_2 \subset \mathcal{P}(\Theta)$  we obtain that  $\mathcal{M}_2$  is finite dimensional. By (4.7),  $\Theta_1 \mathcal{M}_2 \subset \mathcal{P}(\Theta) \subset \mathbf{H}_{2,J}$ . From  $\Theta_1 \mathcal{M}_2 \perp \mathcal{P}(\Theta_1)$  and again by (4.7), we have that  $\Theta_1 \mathcal{M}_2 \subset \Theta_1 \mathbf{H}_{2,J}$ , that is,  $\mathcal{M}_2 \subset \mathbf{H}_{2,J}$ . Let  $\mathbf{f} \in \mathcal{M}_2$ , then  $R_0 \mathbf{f} \in \mathbf{H}_{2,J}$  and  $\Theta_1 R_0 \mathbf{f} \in \Theta_1 \mathbf{H}_{2,J}$ , hence  $\Theta_1 R_0 \mathbf{f} \perp \mathcal{P}(\Theta_1)$ . The equality

$$\Theta_1(z) R_0 \mathbf{f}(z) = R_0(\Theta_1 \mathbf{f})(z) - (R_0 \Theta_1)(z) \mathbf{f}(0)$$

implies that also  $\Theta_1 R_0 \mathbf{f} \in \mathcal{P}(\Theta)$ . Indeed,  $\Theta_1 \mathbf{f} \in \Theta_1 \mathcal{M}_2 \subset \mathcal{P}(\Theta)$ , so  $R_0(\Theta_1 \mathbf{f}) \in R_0(\mathcal{P}(\Theta)) \subset \mathcal{P}(\Theta)$ , and, by Theorem 4.2 (1f),  $(R_0 \Theta_1)(z) \mathbf{f}(0) \in \mathcal{P}(\Theta_1) \subset \mathcal{P}(\Theta)$ . Hence  $\Theta_1 R_0 \mathbf{f} \in \mathcal{P}(\Theta) \ominus \mathcal{P}(\Theta_1) = \Theta_1 \mathcal{M}_2$ , that is,  $R_0 \mathbf{f} \in \mathcal{M}_2$  and  $\mathcal{M}_2$  is  $R_0$ -invariant. By Remark 4.3, an eigenfunction  $\mathbf{f}_0 \in \mathcal{M}_2$  of  $R_0$  corresponding to the eigenvalue  $\lambda_0$  has the form

$$\mathbf{f}_0(z) = \frac{\mathbf{c}}{1 - \lambda_0 z}, \quad \mathbf{c} \in \mathbb{C}^2, \mathbf{c} \neq 0.$$

Since  $\Theta_1 \mathbf{f} \in \mathcal{P}(\Theta)$  and by Theorem 4.2 (1d),  $\Theta_1 \mathbf{f}_0$  is a polynomial and hence  $\lambda_0 = 0$ . Indeed,  $\lambda_0 \neq 0$  implies that  $\Theta_1(1/\lambda_0) \mathbf{c} = 0$ , so  $\det \Theta_1(1/\lambda_0) = 0$ , a contradiction. Thus  $\mathcal{M}_2$  consists only of eigenvectors and associated vectors of  $R_0$  corresponding to the eigenvalue 0, and these vectors are polynomials in  $z$ . We have shown that the assumptions of Theorem 4.2 (2) are satisfied, and hence  $\mathcal{M}_2$  is some  $\mathcal{P}(\Theta_2)$  space with  $\Theta_2 \in \mathcal{U}_J$ . The converse follows from Proposition 4.5.  $\square$

## 5. Elementary $J$ -unitary polynomials

Let  $\mathcal{M}$  be a finite-dimensional, say  $N$ -dimensional space of  $\mathbb{C}^2$ -valued polynomials which is  $R_0$ -invariant. It has a basis which consists of the columns of a matrix-function of the form

$$C(I_N - zA)^{-1},$$

where  $(C, A) \in \mathbb{C}^{2 \times N} \times \mathbb{C}^{N \times N}$  and  $A$  is nilpotent (see [16]). The linear independence of the columns of  $C(I_N - zA)^{-1}$  implies that  $\bigcap_0^\infty \ker CA^\ell = \{0\}$ . If  $\mathcal{M}$  is non-degenerate in the  $\mathbf{H}_{2,J}$ -inner product then the Gram matrix  $\mathbb{Q}$  of  $\mathcal{M}$  with respect to the basis defined by the columns of  $C(I - zA)^{-1}$ , which is given by

$$\mathbf{y}^* \mathbb{Q} \mathbf{x} = \langle C(I_N - zA)^{-1} \mathbf{x}, C(I_N - zA)^{-1} \mathbf{y} \rangle_{\mathbf{H}_{2,J}}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^2, \quad (5.1)$$

is invertible. It is readily seen that  $\mathbb{Q}$  is the same as in (4.4), and that it is the unique solution of the matrix equation

$$\mathbb{Q} - A^* \mathbb{Q} A = C^* J C.$$

A function  $\Theta_{w_0} \in \mathcal{U}_J$ , where  $w_0 \in \mathbb{T}$ , such that  $\mathcal{M} = \mathcal{P}(\Theta_{w_0})$  is given by (4.5):

$$\Theta_{w_0}(z) = I_2 - (1 - zw_0^*)C(I_N - zA)^{-1}\mathbb{Q}^{-1}(I_N - w_0^*A^*)^{-1}C^*J. \quad (5.2)$$

This follows from the equality

$$C(I_N - zA)^{-1}\mathbb{Q}^{-1}(I_N - wA)^{-*}C^* = \frac{J - \Theta_{w_0}(z)J\Theta_{w_0}(w)^*}{1 - zw^*}, \quad (5.3)$$

which holds (see [16]) since both sides define a reproducing kernel for  $\mathcal{M}$  and hence coincide. The left hand side is a reproducing kernel for  $\mathcal{M}$  because of the inner product (5.1) and the right hand side by Theorem 4.2 (1a).

The case  $\dim \mathcal{M} = 1$  and  $A = 0$  is of particular interest. Then  $C$  is a vector,  $C = \mathbf{u} \in \mathbb{C}^2$ , and  $\mathbb{Q} = C^* J C = \mathbf{u}^* J \mathbf{u}$ . The function  $\Theta = \theta_1$  corresponding to the choice  $w_0 = 1$  is then equal to

$$\theta_1(z) = I_2 - (1 - z) \frac{\mathbf{u} \mathbf{u}^* J}{\mathbf{u}^* J \mathbf{u}}. \quad (5.4)$$

If  $\mathbf{u}^* J \mathbf{u} > 0$  and  $P = (\mathbf{u} \mathbf{u}^* J) / (\mathbf{u}^* J \mathbf{u})$ , then  $P$  is a projection in  $\mathbb{C}^2$  and  $\theta_1(z) = I_2 - P + zP$  is a Blaschke factor at the point 0 (see [16, (1.2.8)]).

**Proposition 5.1.** *If  $\Theta \in \mathcal{U}_J$  is not constant it admits a minimal (possibly trivial) factorization  $\Theta = \theta_1 \theta_2$  where  $\theta_1$  is of one of the two forms:*

- (a)  $\theta_1(z) = I_2 - (1 - z)((\mathbf{u} \mathbf{u}^* J) / (\mathbf{u}^* J \mathbf{u}))$  with a vector  $\mathbf{u} \in \mathbb{C}^2$  such that  $\mathbf{u}^* J \mathbf{u} \neq 0$ .
- (b)  $\theta_1(z) = z^k I_2 + \mathbf{u} \mathbf{u}^* J Q(z)$  with  $k \geq 1$ , a  $J$ -neutral non-zero vector  $\mathbf{u} \in \mathbb{C}^2$  and a polynomial  $Q(z)$  of degree  $2k$  with the property

$$z^{2k} Q(1/z^*)^* = -Q(z). \quad (5.5)$$

In the proof of the proposition we use the following lemma.

**Lemma 5.2.** *If  $f$  is a polynomial of degree  $k - 1$  then the function  $Q(z) = f(z) - z^{2k} f(1/z^*)^*$  has at least one zero on  $\mathbb{T}$ .*



**Proof.** The function  $g(z) = z^k f(1/z)$  is a polynomial of degree  $k$  and  $g(0) = 0$ . Moreover  $Q(z) = z^k(g(1/z) - g(z^*)^*)$ . If  $Q(z)$  would be different from zero on  $\mathbb{T}$  then  $\operatorname{Im} g(z)$  would not change sign there, say  $\operatorname{Im} g(z) > 0$  for  $|z| = 1$ . Consider the entire function  $h(z) = \exp\{ig(z)\}$ . Then  $|h(z)| = \exp\{-\operatorname{Im} g(z)\} < 1$  on  $\mathbb{T}$ . On the other hand,  $h(0) = 1$ , in contradiction with the maximum modulus principle.  $\square$

**Proof of Proposition 5.1.** We consider the finite dimensional reproducing kernel Pontryagin space  $\mathcal{P}(\Theta)$  associated with  $\Theta$ . Since, by Theorem 4.2(1d) and (1e),  $\mathcal{P}(\Theta)$  is  $R_0$ -invariant and consists of polynomial functions, according to Remark 4.3, it contains a vector

$$\mathbf{u} = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} \in \mathbb{C}^2$$

corresponding to the eigenvalue zero of  $R_0$ . The  $\mathcal{P}(\Theta)$  inner product of  $\mathbf{u}$  with itself is the  $\mathbf{H}_{2,J}$  inner product of  $\mathbf{u}$  with itself, that is,

$$\langle \mathbf{u}, \mathbf{u} \rangle_{\mathcal{P}(\Theta)} = \mathbf{u}^* J \mathbf{u}.$$

If this number is  $\neq 0$ , then the space  $\mathcal{M}_0 := \operatorname{span}\{\mathbf{u}\}$  is non-degenerate and a space  $\mathcal{P}(\theta_1)$  for some  $\theta_1 \in \mathcal{U}_J$ . Moreover, its reproducing kernel is of the form  $\mathbf{u}(\mathbf{u}^* J \mathbf{u})^{-1} \mathbf{u}^*$ , and so, since the reproducing kernel is unique for the space it reproduces,

$$\mathbf{u}(\mathbf{u}^* J \mathbf{u})^{-1} \mathbf{u}^* = \frac{J - \theta_1(z) J \theta_1(w)^*}{1 - zw^*},$$

where  $\theta_1$  (up to a  $J$ -unitary constant matrix) is given by (5.4), and case (a) follows from Theorem 4.6.

Assume now that  $\mathbf{u} \in \ker R_0$  is a neutral vector. If there would be a second linearly independent vector in  $\ker R_0$ , it would also be constant and  $\ker R_0 = \mathbb{C}^2$ , which is non-degenerate in  $\mathbf{H}_{2,J}$ . In this case  $\mathcal{P}(\Theta)$  contains also a constant vector which is not  $J$ -neutral, and we are in the situation in which case (a) applies.

Therefore only the case that  $\mathcal{P}(\Theta)$  is spanned by just one Jordan chain of the operator  $R_0$  at zero remains to be considered. If  $\dim \mathcal{P}(\Theta) = n + 1$  this chain has to be of the form

$$\mathbf{f}_0(z) = \begin{pmatrix} \alpha_0^* \\ \beta_0^* \end{pmatrix}, \quad \mathbf{f}_1(z) = z\mathbf{f}_0(z) + \begin{pmatrix} \alpha_1^* \\ \beta_1^* \end{pmatrix}, \quad \dots, \quad \mathbf{f}_n(z) = z\mathbf{f}_{n-1}(z) + \begin{pmatrix} \alpha_n^* \\ \beta_n^* \end{pmatrix}. \quad (5.6)$$

Set

$$\mathcal{M}_j = \operatorname{span}\{\mathbf{f}_0(z), \mathbf{f}_1(z), \dots, \mathbf{f}_{j-1}(z)\}, \quad j = 1, 2, \dots, n+1. \quad (5.7)$$

Then, evidently,

$$\{0\} \subsetneq \mathcal{M}_1 \subsetneq \mathcal{M}_2 \subsetneq \dots \subsetneq \mathcal{M}_{n+1} = \mathcal{P}(\Theta).$$

Since  $\mathcal{M}_1$  is degenerate, we have  $\mathbf{f}_0^* J \mathbf{f}_0 = 0$  and  $\mathbf{f}_0 \neq 0$  or, equivalently,  $|\alpha_0| = |\beta_0| \neq 0$ . Without loss of generality we may assume  $\alpha_0 = 1$ , and then from the chain (5.6) a basis of  $\mathcal{P}(\Theta)$  can be constructed which is of the form

$$\mathbf{g}_0(z) = \begin{pmatrix} 1 \\ \sigma_0^* \end{pmatrix}, \quad \mathbf{g}_1(z) = z\mathbf{g}_0(z) + \begin{pmatrix} 0 \\ \sigma_1^* \end{pmatrix}, \dots, \quad \mathbf{g}_n(z) = z\mathbf{g}_{n-1}(z) + \begin{pmatrix} 0 \\ \sigma_n^* \end{pmatrix}. \quad (5.8)$$

Let  $k \geq 1$  be the smallest integer for which  $\sigma_k \neq 0$ . If  $\Delta$  is the  $k \times k$  matrix

$$\Delta := \begin{pmatrix} \sigma_k & 0 & \cdots & 0 \\ \sigma_{k+1} & \sigma_k & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{2k-1} & \cdots & \sigma_{k+1} & \sigma_k \end{pmatrix},$$

then  $\Delta$  is invertible and the Gram matrix  $\mathcal{G}_{2k}$  associated with the first  $2k$  elements of the basis (5.8) is given by

$$\mathcal{G}_{2k} = \begin{pmatrix} 0 & -\sigma_0 \Delta^* \\ -\sigma_0^* \Delta & -\Delta \Delta^* \end{pmatrix}.$$

As the first  $k$  elements span a neutral subspace and  $\mathcal{G}_{2k}$  is invertible,  $\mathcal{M}_{2k}$  is the first non-degenerate subspace in the chain (5.7). According to Theorem 4.2 (2), it is a space  $\mathcal{P}(\widehat{\Theta})$  for some  $\widehat{\Theta} \in \mathcal{U}_J$  and by Theorem 4.6,  $\widehat{\Theta}$  is a divisor of  $\Theta$  in  $\mathcal{U}_J$ . In order to calculate  $\widehat{\Theta}$  from formula (5.2) we introduce the  $2k \times 2k$  matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and the  $2 \times 2k$  matrix  $C = (C_0 \ C_1)$  with

$$C_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \sigma_0^* & 0 & \cdots & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sigma_k^* & \sigma_{k+1}^* & \cdots & \sigma_{2k-1}^* \end{pmatrix}.$$

Then the columns of the matrix polynomial  $C(I - zA)^{-1}$  coincide with the basis elements in (5.8) and we can choose  $\widehat{\Theta} = \Theta_{w_0}$  given by (5.2) with  $\mathbb{Q} = \mathcal{G}_{2k}$  and some  $w_0 \in \mathbb{T}$ .

We define complex numbers  $c_n$ ,  $n = 0, 1, \dots$ , by the relation

$$\left( \sum_{j=k}^{\infty} \sigma_j z^j \right) (c_0 + c_1 z + \cdots + c_n z^n + \cdots) = \sigma_0 z^k,$$

and  $k \times k$ -matrices

$$A_1(z) = \begin{pmatrix} 1 & z & \cdots & z^{k-1} \\ 0 & 1 & \cdots & z^{k-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad A_2(z) = \begin{pmatrix} z^k & z^{k+1} & \cdots & z^{2k-1} \\ z^{k-1} & z^k & \cdots & z^{2k-2} \\ \vdots & \vdots & & \vdots \\ z & z^2 & \cdots & z^k \end{pmatrix},$$

$$L = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{k-1} & c_{k-2} & \cdots & c_0 \end{pmatrix} = \sigma_0 \Delta^{-1}.$$

Then

$$\mathcal{G}_{2k}^{-1} = \begin{pmatrix} I_k & -L \\ -L^* & 0_k \end{pmatrix},$$

and for arbitrary  $w_0 \in \mathbb{T}$ ,

$$\begin{aligned} \Theta_{w_0}(z) &= I_2 - (1 - zw_0^*)(C_0 \ C_1) \begin{pmatrix} A_1(z) & A_2(z) \\ 0_k & A_1(z) \end{pmatrix} \begin{pmatrix} I_k & -L \\ -L^* & 0_k \end{pmatrix} \\ &\quad \times \begin{pmatrix} A_1(w_0)^* & 0 \\ A_2(w_0)^* & A_1(w_0)^* \end{pmatrix} \begin{pmatrix} C_0^* \\ C_1^* \end{pmatrix} J \\ &= I_2 - (1 - zw_0^*) \{ C_0 A_1 A_1^* C_0^* - C_1 A_1 L^* A_1^* C_0^* - C_0 A_1 L A_1^* C_1^* \\ &\quad - C_0 A_2 L^* A_1^* C_0^* - C_0 A_1 L A_2^* C_0^* \} J. \end{aligned} \quad (5.9)$$

Here and below, for  $j = 1, 2$ , we write  $A_j$  and  $A_j^*$  for  $A_j(z)$  and  $A_j(w_0)^*$ . Straight-forward calculations give

$$\begin{aligned} C_0 A_1 A_1^* C_0^* &= \left( 1 + zw_0^* + z^2 w_0^{*2} + \cdots + z^{k-1} w_0^{*(k-1)} \right) \begin{pmatrix} 1 & \sigma_0 \\ \sigma_0^* & 1 \end{pmatrix}, \\ C_0 A_1 L A_1^* C_1^* &= \left( 1 + zw_0^* + z^2 w_0^{*2} + \cdots + z^{k-1} w_0^{*(k-1)} \right) \begin{pmatrix} 0 & \sigma_0 \\ 0 & 1 \end{pmatrix}, \\ C_1 A_1 L^* A_1^* C_0^* &= \left( 1 + zw_0^* + z^2 w_0^{*2} + \cdots + z^{k-1} w_0^{*(k-1)} \right) \begin{pmatrix} 0 & 0 \\ \sigma_0^* & 1 \end{pmatrix}, \end{aligned}$$

and, finally,

$$\begin{aligned} &(1 - zw_0^*)(C_0 A_2 L^* A_1^* C_0^* + C_0 A_1 L A_2^* C_0^*) \\ &= (w_0^{*k} Q(z) + z^k Q(w_0)^*) \begin{pmatrix} 1 & \sigma_0 \\ \sigma_0^* & 1 \end{pmatrix}, \end{aligned}$$

where  $Q(z) = c_0 + \cdots + c_{k-1} z^{k-1} - (c_{k-1}^* z^{k+1} + \cdots + c_0^* z^{2k})$ . Applying Lemma 5.2, we can choose  $w_0 \in \mathbb{T}$  such that  $Q(w_0) = 0$ , and inserting the latter expressions into (5.9) we get

$$\Theta_{w_0}(z) = w_0^{*k} \begin{pmatrix} z^k + Q(z) & -\sigma_0 Q(z) \\ \sigma_0^* Q(z) & z^k - Q(z) \end{pmatrix}, \quad (5.10)$$

which is of the same form as in (b).  $\square$

It is interesting to look at special cases of  $\Theta$  in Proposition 5.1(a). If

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we obtain

$$\theta_1(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

while the case

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

leads to

$$\theta_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

If we multiply  $\theta_1(z)$  in Proposition 5.1(a) by a constant  $J$ -unitary matrix from the right, depending on the sign of  $\mathbf{u}^* J \mathbf{u}$  we arrive at the following forms of an elementary  $J$ -unitary polynomial matrix function:

$$\begin{aligned} (a_1) \quad \theta(z) &= \frac{1}{\sqrt{1-|\sigma_0|^2}} \begin{pmatrix} z & \sigma_0 \\ z\sigma_0^* & 1 \end{pmatrix} \quad \text{with } |\sigma_0| < 1, \\ (a_2) \quad \theta(z) &= \frac{1}{\sqrt{|\sigma_0|^2-1}} \begin{pmatrix} \sigma_0 & z \\ 1 & z\sigma_0^* \end{pmatrix} \quad \text{with } |\sigma_0| > 1. \end{aligned}$$

In the case (b), from (5.5) it follows that the polynomial  $Q(z)$  is of the form

$$Q(z) = P(z) - z^{2k} P(1/z^*)^* + ic'_k z^k$$

where  $P(z)$  is a polynomial of degree  $k-1$  and the coefficient  $c'_k$  is real. If we multiply  $\theta_1(z)$  in (b) by the  $J$ -unitary matrix  $I + ic'_k \mathbf{u} \mathbf{u}^* J$  from the right we obtain a new function  $\theta(z)$  of the same form where now  $c'_k = 0$ .

There is little to add to the proof of Proposition 5.1 to get the following result.

**Theorem 5.3.** *The functions in (a) and in (b) of Proposition 5.1 are (up to constant  $J$ -unitary factors from the right) the only elementary polynomials in  $\mathcal{U}_J$ .*

**Proof.** Let  $\theta \in \mathcal{U}_J$  be elementary. Then the corresponding space  $\mathcal{P}(\theta)$  is spanned by only one chain of the operator  $R_0$ . If the first element  $\mathbf{f}_0$  of this chain is non-degenerate we are in case (a). Otherwise we can suppose that a basis of  $\mathcal{P}(\theta)$  is of the form (5.8) and, by Proposition 5.1, that

$$\text{span}\{\mathbf{g}_0, \dots, \mathbf{g}_{2k}\} = \mathcal{P}(\Theta_{w_0}),$$

where  $\Theta_{w_0}$  is given by (5.10). By the uniqueness (up to a constant  $J$ -unitary factor from the right) of the element of  $\mathcal{U}_J$  associated to a given space we find  $\theta = \Theta_{w_0}U$  where  $U$  is a  $J$ -unitary matrix.  $\square$

The following theorem, which is the main result of this section, generalizes the well known factorization of  $J$ -inner polynomials into a product of factors of the form

$$\frac{1}{\sqrt{1-|\rho|^2}} \begin{pmatrix} 1 & \rho \\ \rho^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

and a  $J$ -unitary constant (see [19]).

**Theorem 5.4.** *Every  $\Theta \in \mathcal{U}_J$  can be written in a unique way as*

$$\Theta(z) = z^n \theta_1(z) \cdots \theta_N(z) U,$$

where  $n$  is a natural number, the  $\theta_j$  are elementary factors normalized by  $\theta_j(1) = I_2$ ,  $U = \Theta(1)$ , and

$$\text{ind}_- \mathcal{P}(\Theta) = n + \sum_{j=1}^N \text{ind}_- \mathcal{P}(\theta_j). \quad (5.11)$$

Moreover,  $n > 0$  if and only if  $\mathcal{P}(\Theta)$  contains more than one chain (for  $R_0$  at the eigenvalue 0) or, equivalently, if and only if

$$\mathbb{C}^2 \subset \mathcal{P}(\Theta).$$

Finally,

$$\mathcal{P}(\Theta) = \mathcal{P}(z^n I_2) \oplus z^n \mathcal{P}(\theta_1 \cdots \theta_N). \quad (5.12)$$

**Proof.** Without loss of generality we may assume that  $\Theta(1) = I_2$ . We write  $\Theta(z) = z^n \tilde{\Theta}(z)$  where  $\tilde{\Theta}(0) \neq 0$ . Then by Example 4.4 and Theorem 4.6,  $n > 0$  if and only if  $\mathbb{C}^2 \subset \mathcal{P}(\Theta)$ . According to the proof of Proposition 5.1, this holds if and only if  $\mathcal{P}(\Theta)$  contains more than one chain and we have

$$\mathcal{P}(\Theta) = \mathcal{P}(z^n I_2) \oplus z^n \mathcal{P}(\tilde{\Theta}). \quad (5.13)$$

Since  $\tilde{\Theta}(0) \neq 0$ , the space  $\mathcal{P}(\tilde{\Theta})$  is spanned by only one chain, say of the form (5.6), and we have a unique increasing sequence

$$\{0\} \subsetneq \mathcal{M}_1 \subsetneq \mathcal{M}_2 \subsetneq \cdots \subsetneq \mathcal{M}_d = \mathcal{P}(\tilde{\Theta})$$

of  $R_0$ -invariant subspaces, where  $\dim \mathcal{M}_j = j$  and  $d = \dim \mathcal{P}(\tilde{\Theta})$ . Consider all indices  $j \in \{1, 2, \dots, d\}$  for which  $\mathcal{M}_j$  is non-degenerate in the  $\mathbf{H}_{2,j}$  inner product and order them:  $j_1 < j_2 < \cdots < j_N = d$ . These indices are defined in a unique way since the subspaces  $\mathcal{M}_j$  form an increasing chain. If  $\Theta_k \in \mathcal{U}_J$  is defined by the relations  $\mathcal{M}_{j_k} = \mathcal{P}(\Theta_k)$  and  $\Theta_k(1) = I_2$ , we have

$$\{0\} \subsetneq \mathcal{P}(\Theta_1) \subsetneq \mathcal{P}(\Theta_2) \subsetneq \cdots \subsetneq \mathcal{P}(\Theta_N) = \mathcal{P}(\tilde{\Theta}).$$

With  $\theta_1 := \Theta_1$  and  $\theta_{k+1} := \Theta_k^{-1} \Theta_{k+1}$ ,  $k = 1, 2, \dots, N-1$ , it follows that

$$\Theta_k = \theta_1 \cdots \theta_k, \quad \tilde{\Theta} = \theta_1 \theta_2 \cdots \theta_N.$$

We claim that the factors  $\theta_k$  are elementary. This is clear for  $\theta_1$ . Now assume that for some  $k > 1$  we have a non-trivial  $J$ -unitary factorization  $\theta_k = \theta_{k1} \theta_{k2}$ . Then the space  $\mathcal{P}(\theta_1 \cdots \theta_{k-1} \theta_{k1})$  is non-degenerate,  $R_0$ -invariant, and such that

$$\mathcal{P}(\Theta_{k-1}) \subsetneq \mathcal{P}(\theta_1 \cdots \theta_{k-1} \theta_{k1}) \subsetneq \mathcal{P}(\Theta_k).$$

This contradicts the fact that the  $\mathcal{M}_{j_k}$  are the only non-degenerate subspaces of the chain.

Finally (5.11) and (5.12) follow from (5.13) and by repeated application of Proposition 4.5.  $\square$

## 6. The reproducing kernel Pontryagin spaces associated with the Schur transformation

Throughout this section  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$ . Then  $u, v \in \mathbf{H}_2$ , and  $|u(0)| + |v(0)| \neq 0$ . We write

$$u(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \quad v(z) = \sum_{n=0}^{\infty} \beta_n z^n,$$

so that  $|\alpha_0| + |\beta_0| \neq 0$ . By  $\mathcal{M}(u, v)$  we denote the linear subset of  $\mathbf{H}_{2,J}$  spanned by the vector functions

$$\mathbf{f}_0(z) = \begin{pmatrix} \alpha_0^* \\ \beta_0^* \end{pmatrix}, \quad \mathbf{f}_1(z) = z\mathbf{f}_0(z) + \begin{pmatrix} \alpha_1^* \\ \beta_1^* \end{pmatrix}, \dots, \quad \mathbf{f}_n(z) = z\mathbf{f}_{n-1}(z) + \begin{pmatrix} \alpha_n^* \\ \beta_n^* \end{pmatrix}, \dots \quad (6.1)$$

The following proposition shows that the linear space  $\mathcal{M}(u, v)$  is isomorphic to a dense linear subset of the space  $\mathcal{P}(u, v)$ . Recall (see (3.2)) that  $\mathcal{P}(u, v)$  is the reproducing kernel Pontryagin space with reproducing kernel

$$S_{(u,v)}(z, w) = \frac{u(z)u(w)^* - v(z)v(w)^*}{1 - zw^*}.$$

**Proposition 6.1.** *If  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$  then the mapping*

$$\mathbf{f} \mapsto (u - v)\mathbf{f}, \quad \mathbf{f} \in \mathcal{M}(u, v), \quad (6.2)$$

*establishes an isometry between the linear space  $\mathcal{M}(u, v)$  equipped with the  $\mathbf{H}_{2,J}$  inner product and a dense linear subset of  $\mathcal{P}(u, v)$ .*

**Proof.** If  $K(z, w)$  is a kernel with a finite number of negative squares and analytic in  $z$  and  $w^*$  in an open subset  $\Omega$  of  $\mathbb{C}$ , and  $\mathcal{P}(K)$  denotes the corresponding reproducing kernel Pontryagin space, then the following holds: For  $n \in \mathbb{N}$  and  $w \in \Omega$  the function

$$z \mapsto \frac{\partial^n K(z, w)}{\partial w^{*n}}, \quad z \in \Omega,$$

belongs to  $\mathcal{P}(K)$ , and for any  $f \in \mathcal{P}(K)$  we have

$$\left\langle f, \frac{\partial^n K(\cdot, w)}{\partial w^{*n}} \right\rangle_{\mathcal{P}(K)} = f^{(n)}(w).$$

It follows that, for  $w \in \Omega$  fixed, the closed linear span of all functions

$$\frac{\partial^n K(\cdot, w)}{\partial w^{*n}}, \quad n \in \mathbb{N},$$

coincides with  $\mathcal{P}(K)$ .

We apply this to the kernel  $S_{(u,v)}(z, w)$  and for this we write

$$S_{(u,v)}(z, w) = (u(z) - v(z)) \frac{\begin{pmatrix} u(w)^* \\ v(w)^* \end{pmatrix}}{1 - zw^*}.$$

We have

$$\begin{aligned} \frac{1}{n!} \frac{\partial^n}{\partial w^{*n}} \frac{\begin{pmatrix} u(w)^* \\ v(w)^* \end{pmatrix}}{1 - zw^*} &= \begin{pmatrix} u(w)^* \\ v(w)^* \end{pmatrix} \frac{z^n}{(1 - zw^*)^{n+1}} + \cdots + \frac{1}{j!} \begin{pmatrix} u(w)^{*(j)} \\ v(w)^{*(j)} \end{pmatrix} \\ &\quad \times \frac{z^{n-j}}{(1 - zw^*)^{n-j+1}} + \cdots + \frac{1}{n!} \begin{pmatrix} u(w)^{*(n)} \\ v(w)^{*(n)} \end{pmatrix} \frac{1}{(1 - zw^*)}, \end{aligned} \quad (6.3)$$

which, for  $w = 0$ , is equal to  $\mathbf{f}_n(z)$ . It follows that for  $n, m \in \mathbb{N}$ ,  $n \geq m$ ,

$$\begin{aligned} &\langle (u - v)\mathbf{f}_n, (u - v)\mathbf{f}_m \rangle_{\mathcal{P}(u,v)} \\ &= \frac{1}{n!m!} \left\langle \frac{\partial^n S_{(u,v)}(\cdot, \zeta)}{\partial \zeta^{*n}}, \frac{\partial^m S_{(u,v)}(\cdot, w)}{\partial w^{*m}} \right\rangle_{\mathcal{P}(u,v)} \Big|_{\zeta=w=0} \\ &= \frac{1}{n!m!} \frac{\partial^{n+m} S_{(u,v)}(w, \zeta)}{\partial \zeta^{*n} \partial w^m} \Big|_{\zeta=w=0} \\ &= \frac{1}{n!m!} \frac{\partial^{n+m}}{\partial \zeta^{*n} \partial w^m} \frac{u(w)u(\zeta)^* - v(w)v(\zeta)^*}{1 - w\zeta^*} \Big|_{\zeta=w=0} \\ &= \sum_{j=0}^m \left( J \begin{pmatrix} \alpha_{m-j}^* \\ \beta_{m-j}^* \end{pmatrix}, \begin{pmatrix} \alpha_{n-j}^* \\ \beta_{n-j}^* \end{pmatrix} \right)_{\mathbb{C}^2}. \end{aligned}$$

On the other hand, the latter expression equals  $[\mathbf{f}_n, \mathbf{f}_m]_{\mathbf{H}_{2,J}}$ . Therefore the mapping (6.2) is an isometry between the total set of all  $\mathbf{f}_n \in \mathcal{M}(u, v)$  and the set of all  $(u - v)\mathbf{f}_n$ ,  $n = 0, 1, \dots$ , which is total in  $\mathcal{P}(u, v)$ .  $\square$

With the pair  $(u, v) \in \mathbf{S}_k^{\text{hom}}$  we associate the sequence of functions defined recursively by (6.1). Let  $\mathcal{M}_N(u, v)$  be the  $N$ -dimensional space spanned by the  $\mathbf{f}_j$ ,  $j = 0, \dots, N - 1$ . This space  $\mathcal{M}_N(u, v)$  is also spanned by the columns of the matrix-function

$$F_N(z) = C_N(I_N - zA_N)^{-1},$$

where

$$C_N = \begin{pmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_{N-1}^* \\ \beta_0^* & \beta_1^* & \cdots & \beta_{N-1}^* \end{pmatrix} \in \mathbb{C}^{2 \times N},$$

$$A_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{N \times N}. \quad (6.4)$$

As a consequence of Theorem 4.2 (2), we have:

**Proposition 6.2.** *If  $\mathcal{M}_N(u, v)$  is non-degenerate in the  $\mathbf{H}_{2,J}$ -inner product, then there exists a  $\Theta_N \in \mathcal{U}_J$ , uniquely determined by the normalization  $\Theta_N(1) = I_2$ , such that  $\mathcal{M}_N(u, v) = \mathcal{P}(\Theta_N)$ .*

The key to the Schur algorithm in the present setting is the following observation.

**Proposition 6.3.** *Assume that the space  $\mathcal{P}(u, v)$  is non-trivial:  $\mathcal{P}(u, v) \neq \{0\}$ . Then there exists a positive integer  $N$  such that the space  $\mathcal{M}_N(u, v)$  is non-degenerate and hence  $\mathcal{M}_N(u, v) = \mathcal{P}(\Theta_N)$  for some  $\Theta_N \in \mathcal{U}_J$ . If  $N$  is chosen minimal then  $\Theta_N$  is an elementary  $J$ -unitary polynomial, and hence  $N = 1$  or  $N = 2k$  for some  $k \in \mathbb{N}$ .*

**Proof.** By Proposition 6.1 and since any dense linear subset of a Pontryagin space contains a maximal negative (hence  $\kappa$ -dimensional) subspace, we obtain that  $\mathcal{M}(u, v) = \bigcup_{N=1}^{\infty} \mathcal{M}_N(u, v)$  contains a  $\kappa$ -dimensional negative subspace. Because the subspaces  $\mathcal{M}_N(u, v)$  are increasing with  $N$  there is an  $N$  such that  $\mathcal{M}_N(u, v)$  contains a  $\kappa$ -dimensional negative subspace and hence is non-degenerate. By Proposition 6.2,  $\mathcal{M}_N(u, v) = \mathcal{P}(\Theta_N)$  for some  $\Theta_N \in \mathcal{U}_J$ . We take  $N$  minimal with respect to this property. From the proof of Proposition 5.1 it follows that:



- (i) if  $\langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathbf{H}_{2,J}} \neq 0$  then  $N = 1$  and  $\Theta_1$  is an elementary polynomial of type (a).  
(ii) if  $\langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathbf{H}_{2,J}} = 0$  then, since  $\mathcal{M}_N(u, v)$  is non-degenerate, there exists a smallest  $k$  such that  $\langle \mathbf{f}_0, \mathbf{f}_k \rangle_{\mathbf{H}_{2,J}} \neq 0$  and then  $N = 2k$  and  $\Theta_{2k}$  is an elementary polynomial of type (b).  $\square$

In the next two results we elaborate on the Schur algorithm for the pair  $(u, v) \in \mathbf{S}_k^{\text{hom}}$  (see (2.5)). In particular, we explain the relation between  $\alpha$  and  $\Theta$  in that formula. We begin with the inverse problem mentioned in Section 1.

**Proposition 6.4.** *Let  $(u, v) \in \mathbf{S}_k^{\text{hom}}$  and let  $\Theta \in \mathcal{U}_J$  be such that  $\Theta(0) \neq 0$  and*

$$(u(z) - v(z))\Theta(z) = z^{\deg \Theta}(\widehat{u}(z) - \widehat{v}(z)), \quad (6.5)$$

where  $\widehat{u}$  and  $\widehat{v}$  are analytic in  $\mathbb{D}$ . Then  $\mathcal{P}(\Theta)$  is spanned by the elements  $\mathbf{f}_0(z), \mathbf{f}_1(z), \dots, \mathbf{f}_{\ell-1}(z)$  from (6.1) with  $\ell = \deg \Theta$ .

**Proof.** We use power series expansions near  $z = 0$ . If

$$\Theta(z) = \sum_{n=0}^{\infty} \Theta_n z^n,$$

then (6.5) implies

$$\sum_{j=0}^n (\alpha_j - \beta_j) \Theta_{n-j} = 0, \quad n = 0, 1, \dots, \ell - 1. \quad (6.6)$$

The polynomials in (6.1) are given by

$$\mathbf{f}_n(z) = \sum_{k=0}^n \begin{pmatrix} \alpha_k^* \\ \beta_k^* \end{pmatrix} z^{n-k}, \quad n = 0, 1, \dots$$

If  $\mathbf{u} \in \mathbf{H}_2^2$  has the expansion  $\mathbf{u}(z) = \sum_{n=0}^{\infty} \mathbf{u}_n z^n$ , then

$$\Theta(z)\mathbf{u}(z) = \sum_{j=0}^{\infty} \left( \sum_{i=0}^j \Theta_{j-i} \mathbf{u}_i \right) z^j,$$

and it follows that

$$\begin{aligned} \langle \Theta \mathbf{u}, \mathbf{f}_n \rangle_{\mathbf{H}_{2,J}} &= \sum_{k=0}^n (\alpha_k - \beta_k) \left( \sum_{i=0}^{n-k} \Theta_{n-i-k} \mathbf{u}_i \right) \\ &= \sum_{i=0}^n \left( \sum_{k=0}^{n-i} (\alpha_k - \beta_k) \Theta_{n-i-k} \right) \mathbf{u}_i = 0. \end{aligned}$$

Hence  $\mathbf{f}_0, \dots, \mathbf{f}_{\ell-1} \in \mathbf{H}_{2,J} \ominus \Theta \mathbf{H}_{2,J} = \mathcal{P}(\Theta)$ . As these polynomials are linearly independent and as, by Theorem 4.2 (1b),  $\dim \mathcal{P}(\Theta) = \deg \Theta$ , they span  $\mathcal{P}(\Theta)$ .  $\square$

We note that the arguments in the proof above show that the condition (6.5) already implies that the map  $\mathbf{f} \rightarrow (u - v)\mathbf{f}$  is an isometry from the span of the  $\mathbf{f}_j(z)$ ,  $j = 0, \dots, \deg \Theta - 1$ , into  $\mathcal{P}(u, v)$ . We also remark that if  $\mathbf{f}_0$  is non-degenerate, then for the elementary polynomial matrix  $\Theta$  we get  $\mathcal{P}(\Theta) = \text{span}\{\mathbf{f}_0\}$  and thus  $\deg \Theta = 1$ . This happens, in particular, in the positive case.

The solution of the inverse problem as formulated in Section 1 is as follows.

**Theorem 6.5.** *Given a pair  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$ . Then there exists a unique (finite or infinite) sequence of elementary factors  $\theta_j \in \mathcal{U}_J$ ,  $j = 1, 2, \dots$ , normalized by  $\theta_j(1) = I_2$ , with the following property: The solutions of the inverse problem are exactly the functions of the form*

$$\Theta(z) = \theta_1(z) \cdots \theta_N(z)V, \quad N = 1, 2, \dots,$$

where  $V$  is a  $J$ -unitary constant.

**Proof.** By Proposition 6.4, if  $\Theta$  is a solution of the inverse problem then the corresponding space  $\mathcal{P}(\Theta)$  is spanned by the chain of length  $\ell = \deg \Theta$  of the elements  $\mathbf{f}_0(z), \mathbf{f}_1(z), \dots, \mathbf{f}_{\ell-1}(z)$  from (6.1) which are derived from the pair  $(u, v)$ . Consequently, if  $\Theta_1$  is another solution and  $\deg \Theta_1 > \deg \Theta$  then  $\mathcal{P}(\Theta) \subset \mathcal{P}(\Theta_1)$  and hence  $\Theta$  divides  $\Theta_1$ . The existence of the unique sequence of elementary polynomials  $\theta_j$  can now be proved as in the proof of Theorem 5.4.  $\square$

For the formulation of the following theorem we use the notation as in the proof of Theorem 5.4. We order the numbers in  $\{N \in \mathbb{N} \mid \mathcal{M}_N(u, v) \text{ is non-degenerate}\}$ :  $1 \leq N_1 < N_2 < \dots$ , and let  $\Theta_j \in \mathcal{U}_J$  be such that  $\mathcal{M}_{N_j} = \mathcal{P}(\Theta_j)$  and  $\Theta_j(1) = I_2$ . By Proposition 6.3,  $\Theta_1$  is elementary; we shall call it the *first elementary polynomial* associated with  $(u, v)$  and write  $\Theta_1 = \theta_1$ . As we have seen in the proof of Theorem 5.4,  $\Theta_j$  divides  $\Theta_{j+1}$  and  $\Theta_j^{-1}\Theta_{j+1}$  is an elementary polynomial in  $\mathcal{U}_J$ .

**Theorem 6.6.** *For  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$  and  $j = 1, 2, \dots$  it holds that*

$$(u(z) - v(z))\Theta_j(z) = z^{\deg \Theta_j}(u_j(z) - v_j(z)), \quad (6.7)$$

where  $(u_j, v_j) \in \mathbf{S}_{\kappa_j}^{\text{hom}}$  with  $\kappa_j = \kappa - \text{ind}_- \mathcal{P}(\Theta_j)$ . Moreover,

$$\mathcal{P}(u, v) = z^{\deg \Theta_j} \mathcal{P}(u_j, v_j) \oplus (u(z) - v(z))\mathcal{P}(\Theta_j). \quad (6.8)$$

The theorem implies the step-by-step Schur algorithm for homogeneous pairs. If  $(u_0, v_0) := (u, v)$  and if  $\theta_j$  is the first elementary polynomial for  $(u_{j-1}, v_{j-1})$ ,  $j = 1, 2, \dots$ , then

$$\begin{aligned}
(u(z) - v(z))\theta_1(z) &= z^{\deg \theta_1} (u_1(z) - v_1(z)) \\
(u_1(z) - v_1(z))\theta_2(z) &= z^{\deg \theta_2} (u_2(z) - v_2(z)) \\
&\vdots \\
(u_{j-1}(z) - v_{j-1}(z))\theta_j(z) &= z^{\deg \theta_j} (u_j(z) - v_j(z)),
\end{aligned} \tag{6.9}$$

so that  $\Theta_j(z) = \theta_1(z) \cdots \theta_j(z)$  and (6.7) holds.

**Proof of Theorem 6.6.** We use mathematical induction. First we prove the theorem for  $j = 1$ . There are two possibilities:

- (i)  $|\alpha_0|^2 - |\beta_0|^2 \neq 0$  and  $\theta_1$  is of the form (a) of Proposition 5.1 with  $\mathbf{u} = \begin{pmatrix} \alpha_0^* \\ \beta_0^* \end{pmatrix}$ .
- (ii)  $|\alpha_0| = |\beta_0| \neq 0$  and  $\theta_1$  is of the form (b) of Proposition 5.1 with  $\mathbf{u} = \begin{pmatrix} 1 \\ \sigma_0^* \end{pmatrix}$  and  $\sigma_0 = (\beta_0/\alpha_0)$ .

Case (i): We have

$$\begin{aligned}
(\tilde{u}(z) - \tilde{v}(z)) &:= (u(z) - v(z))\theta_1(z) \\
&= (u(z) - v(z)) - \frac{1-z}{|\alpha_0|^2 - |\beta_0|^2} (u(z)\alpha_0^* - v(z)\beta_0^*)(\alpha_0 - \beta_0).
\end{aligned}$$

Evidently,  $\tilde{u}(z)$  and  $\tilde{v}(z)$  vanish at  $z = 0$ . The coefficients of  $z$  in  $\tilde{u}(z)$  and  $\tilde{v}(z)$  are given by

$$\tilde{\alpha}_1 = \frac{\alpha_0(|\alpha_0|^2 - |\beta_0|^2) + \beta_0^*(\alpha_0\beta_1 - \alpha_1\beta_0)}{|\alpha_0|^2 - |\beta_0|^2}$$

and

$$\tilde{\beta}_1 = \frac{\beta_0(|\alpha_0|^2 - |\beta_0|^2) + \alpha_0^*(\alpha_0\beta_1 - \alpha_1\beta_0)}{|\alpha_0|^2 - |\beta_0|^2}.$$

Assuming that both expressions are equal to zero, we find that

$$0 = \alpha_0^*\tilde{\alpha}_1 - \beta_0^*\tilde{\beta}_1 = |\alpha_0|^2 - |\beta_0|^2,$$

which contradicts the hypothesis in Case (i). Hence

$$(\tilde{u}(z) - \tilde{v}(z)) = z(u_1(z) - v_1(z)),$$

with  $|u_1(0)| + |v_1(0)| \neq 0$ , that is, (6.7) holds. The equality (6.8) follows from

$$S_{(u,v)}(z, w) = zw^*S_{(u_1,v_1)}(z, w) + (u(z) - v(z))K_{\theta_1}(z, w) \begin{pmatrix} u(w)^* \\ -v(w)^* \end{pmatrix}.$$

The fact that  $\mathbf{f} \mapsto (u - v)\mathbf{f}$  is an isometry from  $\mathcal{P}(\theta_1) = \mathcal{M}_1(u, v)$  into  $\mathcal{P}(u, v)$  implies that the sum on the right hand side of (6.8) is orthogonal. If  $|\alpha_0|^2 - |\beta_0|^2 > 0$

then  $\mathcal{P}(\theta_1)$  is a Hilbert space (of dimension 1), if  $|\alpha_0|^2 - |\beta_0|^2 < 0$  then  $\mathcal{P}(\theta_1)$  is a Pontryagin space (of dimension 1) with  $\text{ind}_- \mathcal{P}(\theta_1) = 1$ . This implies the formula for  $\kappa_1$ .

Case (ii): The function  $u(z)$  does not vanish at the origin. With  $s(z) = v(z)/u(z)$  and  $\sigma_0 = s(0)$  we have that for some  $l \geq 2k$  and some  $t_l \neq 0$ ,

$$Q(z)(\sigma_0^* s(z) - 1) = z^k + t_l z^l + \dots$$

and therefore we can write

$$\begin{aligned} (u(z) - v(z))\theta_1(z) &= u(z)(1 - s(z))\theta_1(z) \\ &= u(z)\{z^k(1 - s(z)) - (\sigma_0^* s(z) - 1)Q(z)(1 - \sigma_0)\} \\ &= u(z)(\tilde{u}(z) - \tilde{v}(z)), \end{aligned}$$

where

$$\begin{aligned} \tilde{u}(z) &= -t_l z^l - \dots, \\ \tilde{v}(z) &= s(z)z^k - \sigma_0(z^k + t_l z^l + \dots) \\ &= z^k(\sigma_0 + \sigma_k z^k + \dots) - \sigma_0(z^k + t_l z^l + \dots) \\ &= \begin{cases} (\sigma_k - \sigma_0 t_l)z^{2k} + \dots & \text{if } l = 2k, \\ \sigma_k z^{2k} + \dots & \text{if } l > 2k, \end{cases} \end{aligned}$$

so that

$$(\tilde{u}(z) - \tilde{v}(z)) = z^{2k}(u_1(z) - v_1(z))$$

and  $|u_1(0)| + |v_1(0)| \neq 0$ , because

$$\begin{cases} u_1(0) = -u(0)t_l \neq 0 & \text{if } l = 2k, \\ v_1(0) = u(0)\sigma_k \neq 0 & \text{if } l > 2k. \end{cases}$$

Formula (6.8) for this case can be proved in the same way as for Case (i). That  $\kappa_1 = \kappa - k$  follows from the fact that the  $2k$  dimensional Pontryagin space  $\mathcal{P}(\theta_1) = \mathcal{M}_1(u, v)$  has negative index  $k$ .

Assume the theorem holds for some  $j \geq 1$ . Then  $(u_j, v_j) \in \mathbf{S}_{\kappa_j}^{\text{hom}}$ . If  $\theta_{j+1}$  is the first elementary polynomial for  $(u_j, v_j)$  then as for the case  $j = 1$

$$\begin{aligned} (u(z) - v(z))\theta_j(z)\theta_{j+1}(z) &= z^{\deg \theta_j} (u_j(z) - v_j(z))\theta_{j+1}(z) \\ &= z^{\deg \theta_j + \deg \theta_{j+1}} (u_{j+1}(z) - v_{j+1}(z)), \end{aligned}$$

where  $(u_{j+1}, v_{j+1}) \in \mathbf{S}_{\kappa_{j+1}}^{\text{hom}}$  with  $\kappa_{j+1} = \kappa_j - \text{ind}_- \mathcal{P}(\theta_{j+1})$ . By Proposition 4.5, the right-hand side can be written as

$$(u(z) - v(z))\theta_j(z)\theta_{j+1}(z) = z^{\deg \theta_j \theta_{j+1}} (u_{j+1}(z) - v_{j+1}(z)).$$

Theorem 6.4 implies that  $\mathcal{P}(\theta_j \theta_{j+1}) = \mathcal{P}(\theta_{j+1})$ . By the uniqueness of the reproducing kernel, this equality implies  $\theta_{j+1}(z) = \theta_j(z)\theta_{j+1}(z)U$  for some constant

$J$ -unitary matrix  $U$ . Because of normalization,  $U = I_2$ . The rest of the proof employs arguments used before and is therefore omitted.  $\square$

**Theorem 6.7.** *If  $(u, v) \in \mathbf{S}_\kappa^{\text{hom}}$ , then for the elementary polynomials  $\theta_1(z), \theta_2(z), \dots$ , appearing in the Schur algorithm (6.9) for  $(u, v)$  the relation*

$$\kappa = \sum_j \text{ind}_- \mathcal{P}(\theta_j)$$

*holds, where the sum runs over all the (finitely or infinitely many) elementary factors appearing in the Schur algorithm. If, in particular,  $\kappa > 0$ , then at least one of the  $\theta_j$ 's is of the form (a) with  $|\alpha_0| < |\beta_0|$  or of the form (b) of Proposition 5.1.*

**Proof.** Let  $N$  be as in the proof of Proposition 6.3. Then  $\mathcal{M}_N(u, v) = \mathcal{P}(\Theta_N)$  contains a  $\kappa$  dimensional negative subspace and so do all non-degenerate subspaces  $\mathcal{M}_k(u, v)$  with  $k \geq N$ . Hence each  $\theta_j$  for which  $\text{ind}_- \mathcal{P}(\theta_j) > 0$  appears as an elementary factor of  $\Theta_N$ . By Theorem 5.4, this implies the equality of the theorem.  $\square$

**Remark 6.8.** The last statement of the theorem can also be proved by contradiction. Assume the opposite. All  $\theta_j$ 's are of the form (a) with  $|\alpha_0| > |\beta_0|$ . Then for all  $N$ ,  $\Theta_N(z)$  is  $J$ -inner and so is  $\tilde{\Theta}_N(z) = \Theta_N(z^*)^*$ . Write

$$\Theta_N(z) = \begin{pmatrix} a_N(z) & b_N(z) \\ c_N(z) & d_N(z) \end{pmatrix}.$$

Then from the 2, 2-entry of the kernel  $K_{\tilde{\Theta}_N}(z, w)$  we obtain that the function

$$z \mapsto \frac{b_N(z)}{d_N(z)}$$

is a classical Schur function on  $\mathbb{D}$ . From (6.7) it follows that

$$\frac{v(z)}{u(z)} - \frac{b_N(z)}{d_N(z)} = z^{\deg \Theta_N} \frac{v_N(z)}{u(z)d_N(z)},$$

which implies that the first  $\deg \Theta_N$  coefficients of  $(v(z)/u(z))$  coincide with those of a classical Schur function. As  $N \rightarrow \infty$  we have  $\deg \Theta_N \rightarrow \infty$  and  $(b_N(z)/d_N(z))$  (or a subsequence) converges to a Schur function. It follows that  $(v(z)/u(z))$  itself is a Schur function, that is,  $(u, v) \in \mathbf{S}_0^{\text{hom}}$ , contradicting the fact that  $\kappa > 0$ .

We now present a practical algorithm to obtain the factorization of an element  $\Theta \in \mathcal{U}_J$ ,

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

into a minimal product of elementary factors in  $\mathcal{U}_J$ . We first note that

$$\det \Theta(z) = cz^{\deg \Theta}$$

for some complex number  $c$  with  $|c| = 1$ . Indeed, this holds for  $J$ -unitary constant matrices (see after Lemma 2.1), for elementary factors (see (2.7)) and then for any  $U \in \mathcal{U}_J$  by the additive property of the degree for minimal factorization. Without loss of generality we may assume that  $\Theta(0) \neq 0$ . For any  $\zeta$  in the unit circle, the pair  $(c(z)\zeta + d(z), a(z)\zeta + b(z))$ , defined by

$$(c(z)\zeta + d(z) \quad -a(z)\zeta - b(z)) = z^{\deg \Theta} (1 \quad -\zeta) \Theta(z)^{-1}$$

is in some class  $S_k^{\text{hom}}$ . We have

$$(c(z)\zeta + d(z) \quad -a(z)\zeta - b(z)) \Theta(z) = (1 \quad -\zeta) z^{\deg \Theta}, \quad (6.10)$$

and therefore by Proposition 6.4 we see that  $\mathcal{P}(\Theta)$  is spanned by a chain of length  $\deg \Theta$  built from the pair  $(c(z)\zeta + d(z), a(z)\zeta + b(z))$ .

Applying the Schur algorithm to the pair  $(c(z)\zeta + d(z), a(z)\zeta + b(z))$ , we obtain an elementary factor  $\theta_1 \in \mathcal{U}_J$  of degree  $N_1$  and such that

$$(c(z)\zeta + d(z) \quad -a(z)\zeta - b(z)) \theta_1(z) = z^{N_1} (u_1(z) \quad -v_1(z)), \quad (6.11)$$

where  $|u_1(0)| + |v_1(0)| \neq 0$ . By Proposition 6.4 we have that  $\mathcal{P}(\theta_1) \subset \mathcal{P}(\Theta)$ . We reiterate and apply the Schur algorithm to the pair  $(u_1(z), v_1(z))$  and obtain an elementary factor  $\theta_2$  of degree  $N_2$  and such that

$$(u_1(z) \quad -v_1(z)) \theta_2(z) = z^{N_2} (u_2(z) \quad -v_2(z)).$$

Multiplying both sides of (6.11) by  $\theta_2$  on the right we obtain

$$(c(z)\zeta + d(z) \quad -a(z)\zeta - b(z)) \theta_1(z) \theta_2(z) = z^{N_1+N_2} (u_2(z) \quad -v_2(z)),$$

and once more applying Proposition 6.4 leads to

$$\mathcal{P}(\theta_1 \theta_2) \subset \mathcal{P}(\Theta).$$

We obtain a sequence  $\theta_1, \dots, \theta_N$  and can reiterate as long as the inclusion

$$\mathcal{P}(\theta_1 \cdots \theta_N) \subset \mathcal{P}(\Theta) \quad (6.12)$$

is strict. When we have equality in (6.12) we obtain the factorization  $\Theta = \theta_1 \cdots \theta_N U$  where  $U$  is a constant  $J$ -unitary matrix. Hence it follows from (6.10) that

$$(c(z)\zeta + d(z) \quad -a(z)\zeta - b(z)) \theta_1(z) \cdots \theta_N(z) = z^{\deg \Theta} (1 \quad -\zeta) U,$$

and the algorithm stops after  $N$  iterations.

## 7. Some examples

We start with a simple example of a function in  $S_1$ .

**Example 7.1.** Let  $s(z) = 1/(1 - 2z)$ . Since

$$\frac{1}{1 - 2z} = \frac{1}{z - 2} \left( \frac{z - 1/2}{1 - z/2} \right)^{-1} = s_0(z) b(z)^{-1},$$

with a function  $s_0 \in \mathbf{S}_0$  and a Blaschke product  $b$  of degree one it follows that  $s \in \mathbf{S}_1$ . Further, if  $|z| < 1/2$  then

$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n.$$

We consider the space  $\mathcal{M}_1$  spanned by the functions

$$f_0(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_1(z) = z \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The corresponding Gram matrix is

$$\mathbb{P} = \begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix}.$$

Formula (5.2) with  $w_0 = 1$ ,  $\mathbb{Q} = \mathbb{P}$ , and

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

gives

$$\Theta_1(z) = I_2 + \frac{1-z}{2} \begin{pmatrix} z-1 & z+1 \\ z+1 & z+3 \end{pmatrix} J = zI_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \ 1) J \frac{1}{2}(1-z^2),$$

which is of the form  $\Theta_b$  from Section 2. The Schur algorithm gives  $s_1(z) = z/(z-2)$ , which, as to be expected, is a Schur function.

In the next example the function  $s$  has a pole at the origin.

**Example 7.2.** Let  $s(z) = s_0(z)/z^\kappa$ , where  $s_0$  is a Schur function which does not vanish at  $z = 0$  and  $\kappa \in \mathbb{N}$ . Then the first  $\kappa$  steps of the Schur algorithm lead to

$$(z^\kappa \ -s_0(z)) \begin{pmatrix} 1 & 0 \\ 0 & z^\kappa \end{pmatrix} = z^\kappa (1 \ -s_0(z)). \quad (7.1)$$

Indeed, consider the pair  $(z^\kappa, s_0(z))$ , which belongs to  $\mathbf{S}_\kappa^{\text{hom}}$ . The Schur algorithm reiterated  $\kappa$  times gives  $\kappa$  times a factor

$$\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

and hence (7.1).

In the following example the function  $s$  is not meromorphic in  $\mathbb{D}$  (and, in particular, it does not belong to the class  $\mathbf{S}$ ), but the kernel (1.1) has just one negative square.

**Example 7.3.** Let  $\omega \in \mathbb{D}$ ,  $\omega \neq 0$ , and consider the function  $s(z) = \delta_\omega(z)$  (see Example 3.3). Then the Schur algorithm gives

$$s_1(z) = \frac{s(z) - s(0)}{z(1 - s(z)s(0)^*)} = \begin{cases} 0 & \text{if } z \neq \omega, \\ \frac{1}{\omega} & \text{if } z = \omega. \end{cases}$$

Thus  $s_1$  satisfies the relation  $s(z) = zs_1(z)$ .

In the last example the function  $s$  is holomorphic and of bounded type in  $\mathbb{D}$ , but the kernel (1.1) has infinitely many positive and infinitely many negative squares.

**Example 7.4.** Let  $s(z) = (1/1 - z)$ . Then for  $N = 1, 2, \dots$ , the Gram matrix of the space  $\mathcal{M}_N(u, v)$  with  $u(z) = 1 - z$  and  $v(z) = 1$  is the  $N \times N$  matrix  $\mathbb{P}_N$  in the left upper corner of the matrix

$$\mathbb{P} = \begin{pmatrix} 0 & -1 & 0 & 0 & \\ -1 & 1 & -1 & 0 & \ddots \\ 0 & -1 & 1 & -1 & \ddots \\ 0 & 0 & -1 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The matrix  $\mathbb{P}_2$  is not singular and we have  $\mathcal{M}_2(u, v) = \mathcal{H}(\Theta_2)$  with

$$\Theta_2(z) = I_2 + (1 - z) \begin{pmatrix} z & 1 + z \\ 1 + z & 2 + z \end{pmatrix} J.$$

Applying the Schur transformation, cases (b) and subsequently  $(a_+)$  of Section 2, and continuing with similar calculations, we obtain

$$s_1(z) = -zs(z), \quad s_2(z) = -s(z), \quad s_3(z) = zs(z), \quad s_4(z) = s(z), \dots$$

## Acknowledgements

This research was supported in part by grants from the Russian Foundation for Basic Research RFBR 02-01-00353, the Netherlands Organization for Scientific Research NWO 047-008-008 and NWO 61-453, and by the Research Training Network HPRN-CT-2000-00116 of the European Union.

## References

- [1] R. Ackner, H. Lev-Ari, T. Kailath, The Schur algorithm for matrix-valued meromorphic functions, *SIAM J. Matrix Anal. Appl.* 15 (1994) 140–150.
- [2] D. Alpay, T. Azizov, A. Dijksma, H. Langer, The Schur algorithm for generalized Schur functions, I: Coisometric realizations, *Operator Theory: Adv. Appl.* 129, Birkhäuser Verlag, Basel, 2001, pp. 1–36.
- [3] D. Alpay, T. Azizov, A. Dijksma, H. Langer, The Schur algorithm for generalized Schur functions, II: Jordan chains and transformations of characteristic functions, *Monatsh. Math.*, in press.



- [4] D. Alpay, T.Ya. Azizov, A. Dijksma, H. Langer, G. Wanjala, A basic interpolation problem for generalized Schur functions and coisometric realizations, *Operator Theory: Adv. Appl.*, Birkhäuser Verlag, Basel, to appear.
- [5] D. Alpay, A. Dijksma, J. Rovnyak, H. de Snoo, Schur functions, operator colligations, and reproducing kernel Pontryagin spaces, *Operator Theory: Adv. Appl.* 96, Birkhäuser Verlag, Basel, 1997.
- [6] D. Alpay, H. Dym, On applications of reproducing kernel spaces to the Schur algorithm and rational  $J$ -unitary factorization, *Operator Theory: Adv. Appl.* 18, Birkhäuser Verlag, Basel, 1986, pp. 89–159.
- [7] D. Alpay, H. Dym, On a new class of reproducing kernel Hilbert spaces and a new generalization of the Iohvidov's laws, *Linear Algebra Appl.* 178 (1993) 109–183.
- [8] D. Alpay, I. Gohberg, Unitary rational matrix functions, *Operator Theory: Adv. Appl.* 33, Birkhäuser Verlag, Basel, 1988, pp. 175–222.
- [9] J. Ball, J.W. Helton, A Beurling–Lax theorem for the Lie group  $U(m, n)$  which contains most classical interpolation theory, *J. Operator Theory* 8 (1983) 107–142.
- [10] H. Bart, I. Gohberg, M. Kaashoek, Minimal factorization of matrix and operator functions, *Operator Theory: Adv. Appl.* 1, Birkhäuser Verlag, Basel, 1979.
- [11] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.-P. Schreiber, *Pisot and Salem Numbers*, Birkhäuser Verlag, Basel, 1992.
- [12] L. de Branges, Some Hilbert spaces of analytic functions I, *Trans. Amer. Math. Soc.* 106 (1963) 445–468.
- [13] C. Chamfy, Fonctions méromorphes sur le cercle unité et leurs séries de Taylor, *Ann. Inst. Fourier* 8 (1958) 211–251.
- [14] J. Dufresnoy, Sur le problème des coefficients par certaines fonctions dans le cercle unité, *Ann. Acad. Sci. Fenn. Ser. A I* 250 (9) (1958) 1–7.
- [15] P. Delsarte, Y. Genin, Y. Kamp, Pseudo-Carathéodory functions and Hermitian Toeplitz matrices, *Philips J. Res.* 41 (1) (1986) 1–54.
- [16] H. Dym,  $J$  contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, *CBMS Regional Conference Series*, Amer. Math. Soc. 71, Providence, RI, 1989.
- [17] M.G. Krein, H. Langer, Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume  $\Pi_{\kappa}$ , *Colloquia Math. Soc. Janos Bolyai, Tihany (Hungary)* 5, Hilbert space operators, 1970, pp. 353–399.
- [18] M.G. Krein, On an integral representation of a continuous Hermitian-indefinite function with a finite number of negative squares, *Dokl. Akad. Nauk. SSSR* 125 (1959) 31–34.
- [19] V.P. Potapov, The multiplicative structure of  $J$ -contractive matrix-functions, *Trudy Moskow. Mat. Obs.* 4 (1955) 125–236 (English transl.: *Amer. Math. Soc. Transl.* 15(2) (1960) 131–243).
- [20] L. Schwartz, Sous espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants), *J. Analyse Math.* 13 (1964) 115–256.
- [21] P. Sorjonen, Pontryagin Räume mit einem reproduzierenden kern, *Ann. Acad. Sci. Fenn. Ser. A I* 594 (1975) 1–30.